

Rate and G- matrices of a non-negative block tri-diagonal matrix and application to a 3-dimensional skip-free Markov modulated random walk

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Abstract

First, we extend matrix analytic methods to the case of nonnegative block tri-diagonal matrices with countably many phase states, where rate matrices and so-called G-matrices are redefined and some properties of them are clarified. Second, we apply the results to a three-dimensional skip-free Markov modulated random walk (3D-MMRW for short) and obtain a lower bound for the directional asymptotic decay rates of each row of the fundamental matrix (occupation measure) arising from the 3D-MMRW.

Keywords: Nonnegative matrix, matrix analytic method, Markov modulated random walk, queueing model, convergence parameter, decay rate

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1 Introduction

In this paper, we develop tools for analyzing asymptotics of queueing models having many queues and demonstrate their effectiveness and imitations. Typical examples of such queueing model are queueing networks and multiqueue models. Applications of queueing models spread over various areas such as production systems, information network systems and certain service systems, and such a system is often modeled by a queueing network or multiqueue model because it consists of several elements (nodes) or contains several kinds of job (customer). Therefore, analysis of queueing models having many queues is important to design and control relevant systems. The process representing the behavior of a queueing model having many queues is essentially multidimensional, and not only exact analysis but also asymptotic analysis for such a process is generally very hard.

Here we give an example and explain our motivation. A multiqueue model, which is also called a polling model, usually consists of one sever and several queues. Customers arrive at one of the queues according to their customer classes. The server goes around the queues in certain order and serves customers there. The service discipline at each queue is determined by how many customers being served in a visit of the server; a k -limited service is a typical example of service discipline. According to an *exhaustive-type* k -limited service, the server serves customers in a queue until the number of customers served in a visit becomes k or the queue becomes empty.

If the arrival processes and service processes of a multiqueue model with k -limited services are given by Markovian processes such as Markovian arrival processes and phase-type services, the behavior of the model can be represented as a multidimensional continuous-time Markov chain (CTMC) denoted by $\{\mathbf{Y}(t)\} = \{(X_1(t), X_2(t), \dots, X_d(t), J(t))\}$, where d is the number of queues, for $k \in \{1, 2, \dots, d\}$, $X_k(t)$ is the number of customers in queue k at time t and $J(t)$ is the composite phase state of the arrival and service processes; the state space of $\{J(t)\}$ is usually finite. Therefore, it is expected that, by analyzing the CTMC $\{\mathbf{Y}(t)\}$, we can see the stationary distribution of the multiqueue model. However, it is well known that multiqueue models with k -limited services defy exact analysis except for some special models; see, for example, References [1, 13, 22] and Surveys [18, 19]. We clarified certain asymptotic properties of the stationary distribution of a two-queue model ($d = 2$) with k -limited services in Ozawa [14], and it is our future work to clarify those for a general multiqueue model with k -limited services. For the purpose, it suffices to clarify asymptotic properties of the stationary distribution of a multidimensional CTMC given in the form of $\{(X_1(t), X_2(t), \dots, X_d(t), J(t))\}$.

By the uniformization technique, we can construct a discrete-time Markov chain (DTMC), denoted by $\{\mathbf{Y}_n\} = \{(X_{1,n}, X_{2,n}, \dots, X_{d,n}, J_n)\}$, that has the same stationary distribution of the original CTMC $\{\mathbf{Y}(t)\} = \{(X_1(t), X_2(t), \dots, X_d(t), J(t))\}$. Hence, hereafter, we consider $\{\mathbf{Y}_n\}$ instead of $\{\mathbf{Y}(t)\}$. Let a vector process $\{\mathbf{X}_n\}$ be defined as $\mathbf{X}_n = (X_1(t), X_2(t), \dots, X_d(t))$ and we call it the level process; we call $\{J_n\}$ the phase process. We assume that the level process is skip free in all directions, which means that, for every $k \in \{1, 2, \dots, d\}$, $X_{k,n}$ takes only values in $\{-1, 0, 1\}$. In a queueing model, if we consider only single arrivals and single services, the level process is usually skip free. We call the DTMC $\{\mathbf{Y}_n\}$ a d -dimensional skip-free Markov modulated reflecting random walk (MMRRW for short); when $d = 2$, we call it a two-dimensional quasi-birth-and-death process (QBD process for short), see Ozawa [14].

As pointed out in Miyazawa [10], there are several approaches to attack an asymptotic problem for the stationary distribution of such a multidimensional process. Here we take a Markov additive approach based on Matrix analytic methods in queueing theory. The research field of matrix analytic method was originated by M.F. Neuts and it has been providing many algorithms to compute the stationary distributions and related performance measures for various queueing models (see, for example, Neuts [12] and Latouche and Ramaswami [6]). Matrix analytic methods are also used for analyzing asymptotics of the stationary distributions of queueing models including those having two queues (see, for example, He et al. [3], Miyazawa and Zhao[8], Miyazawa [9, 11], Ozawa [14] and Takahashi et al. [20]). In such literature, the behavior of a queueing model is represented as a QBD process with countably many phase state, whose transition probability matrix P is given in tri-diagonal block form as

$$P = \begin{pmatrix} B_0 & B_1 & & & \\ B_{-1} & A_0 & A_1 & & \\ & A_{-1} & A_0 & A_1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad (1.1)$$

where the dimensions of the block matrices are countably infinite. Like QBD processes having finite phase states, such a QBD process also have the stationary distribution $\boldsymbol{\pi} = (\boldsymbol{\pi}_k)$ given in block form as

$$\boldsymbol{\pi}_k = \boldsymbol{\pi}_1 R^{k-1}, \quad k \geq 1, \quad (1.2)$$

where R is the rate matrix determined by the triplet (A_{-1}, A_0, A_1) (see, for example, Tweedie [21]). Hence, we can investigate asymptotics of the stationary distribution through the rate matrix R , where the following formula is used as a key formula (see, Lemma 2.2 of He et al. [3] and Lemma 2.3 of Miyazawa [9]):

$$\text{cp}(R) = \sup\{z > 0; \mathbf{x}A_*(z) \leq \mathbf{x}, \mathbf{x} > \mathbf{0}^\top\}; \quad (1.3)$$

$\text{cp}(R)$ is the convergence parameter of R , $\mathbf{0}$ is a column vector of 0's and $A_*(z)$ is given by

$$A_*(z) = z^{-1}A_{-1} + A_0 + zA_1. \quad (1.4)$$

A skip-free MMRRW $\{\mathbf{Y}_n\} = \{(X_{1,n}, X_{2,n}, \dots, X_{d,n}, J_n)\}$ introduced above can also be represented as d kinds of QBD process with countably many phase states, for example, one is $\{\mathbf{Y}_n^{(1)}\} = \{(X_{1,n}, (X_{2,n}, \dots, X_{d,n}, J_n))\}$, where $X_{1,n}$ is the level and $(X_{2,n}, \dots, X_{d,n}, J_n)$ is the phase state.

Here we consider $\{\mathbf{Y}^{(1)}\}$ and focus on formula (1.3). Inequality $\mathbf{x}A_*(z) \leq \mathbf{x}$ for $\mathbf{x} > \mathbf{0}^\top$ implies that $\text{cp}(A_*(z)) \geq 1$ and we obtain

$$\text{cp}(R) = \sup\{z > 0; \text{cp}(A_*(z)) \geq 1\}. \quad (1.5)$$

Hence, in order to obtain $\text{cp}(R)$, it suffice to consider $\text{cp}(A_*(z))$ for $z > 0$. Since the phase state of $\{\mathbf{Y}^{(1)}\}$ is given by $(X_{2,n}, \dots, X_{d,n}, J_n)$, A_{-1} , A_0 and A_1 can be represented in block tri-diagonal form, and hence, $A_*(z)$ can also be represented in block tri-diagonal form, say,

$$A_*(z) = \begin{pmatrix} B_{*,0}(z) & B_1(z) & & & \\ B_{*,-1}(z) & A_{*,0}(z) & A_{*,1}(z) & & \\ & A_{*,-1}(z) & A_{*,0}(z) & A_{*,1}(z) & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \quad (1.6)$$

This $A_*(z)$ is a nonnegative block tri-diagonal matrix, but it may no longer be stochastic or substochastic. Therefore, in order to apply matrix analytic methods to $A_*(z)$, we must extend them to nonnegative block tri-diagonal matrices with countably many phase states. *For the case in which $B_{*,-1}(z) = A_{*,-1}(z)$, $B_0(z) = A_0(z)$ and $B_1(z) = A_1(z)$, we do it in Section 2*, where the rate matrix and so-called G-matrix of a general nonnegative block tri-diagonal matrix are defined and their properties are clarified. If $A_{*,-1}(z)$, $A_{*,0}(z)$ and $A_{*,1}(z)$ are also represented in block tri-diagonal form, our approach can recursively be applied. Here we note that Kijima [4] and Li and Zhao [7] discussed the case where a matrix corresponding to $A_*(z)$ was a substochastic matrix with finite phase states. Our results are also an extension of their results.

In Section 3, we deal with a skip-free three-dimensional Markov modulated random walk (3D-MMRW for short) on the state space $\mathbb{Z}^3 \times S_0$, where \mathbb{Z} is the set of all integers and S_0 is the set of phase states, and focus on a fundamental matrix arising from the 3D-MMRW. The 3D-MMRW is obtained from a 3D-MMRRW by removing the boundaries. We consider a truncated transition probability matrix Q of the 3D-MMRW, obtained by restricting the state space to the nonnegative area $\mathbb{Z}_+^3 \times S_0$, where \mathbb{Z}_+ is the set of all nonnegative integers; the fundamental matrix \tilde{Q} that we focus on is defined as $\tilde{Q} = \sum_{n=0}^{\infty} Q^n$. Q can be represented in block tri-diagonal form like formula (1.1), but $B_{-1} = A_{-1}$, $B_0 = A_0$ and $B_1 = A_1$. In a discrete-time queueing model having many queues, each element of the fundamental matrix \tilde{Q} is the expected number of visits to a state of the queueing model until at least one of the queues becomes empty for the first time. In that

sense, each of the rows of \tilde{Q} is also called an occupation measure. The occupation measure is not a stochastic measure (distribution), but analysis of the occupation measure gives useful insights into analysis of the stationary distributions of multidimensional MMRRWs. $A_*(z)$ obtained from Q is also represented in block tri-diagonal form and hence, we can analyze asymptotics of the fundamental matrix \tilde{Q} by using the results obtained in Section 2. As a result, we get an lower bound for the directional asymptotic decay rate of each row of \tilde{Q} ; this result can easily be extended to the case where the dimension of the model is greater than 3. As an example, a three queue model with 1-limited services is discussed in Section 3. Section 4 concludes the paper.

2 Nonnegative block tri-diagonal matrix and its properties

We introduce some notations. \mathbb{R} is the set of all real numbers and \mathbb{R}_+ that of all nonnegative real numbers; \mathbb{Z} and \mathbb{Z}_+ are already defined. \mathbb{N} is the set of all positive integers and, for $n \geq 1$, \mathbb{N}_n is the set of positive integers less than or equal to n , i.e., $\mathbb{N}_n = \{1, 2, \dots, n\}$. A set \mathbb{H} is defined as $\mathbb{H} = \{-1, 0, 1\}$. For a matrix A , we denote by $[A]_{i,j}$ the (i, j) -element of A . The transpose of a matrix A is denoted by A^\top . The convergence parameter of a square matrix A with a finite or countable dimension is denoted by $\text{cp}(A)$, i.e., $\text{cp}(A) = \sup\{z \in \mathbb{R}_+; \sum_{n=0}^{\infty} z^n A^n < \infty\}$. We denote by $\text{spr}(A)$ the spectral radius of A , which is the maximum modulus of eigenvalue of A . O is a matrix of 0's, e is a column vector of 1's and $\mathbf{0}$ is a column vector of 0's; their dimensions, which are finite or countably infinite, are determined in context. I is the identity matrix.

Consider a nonnegative block tri-diagonal matrix Q and P defined as

$$Q = \begin{pmatrix} A_0 & A_1 & & & \\ A_{-1} & A_0 & A_1 & & \\ & A_{-1} & A_0 & A_1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad P = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ & A_{-1} & A_0 & A_1 & \\ & & A_{-1} & A_0 & A_1 \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where A_{-1} , A_0 and A_1 are nonnegative square matrices with a countable dimension, i.e., for $k \in \mathbb{H}$, $A_k = (a_{k,i,j}, i, j \in \mathbb{Z}_+)$ and every $a_{k,i,j}$ is nonnegative. Q is a truncation of P . We define a matrix A_* as

$$A_* = A_{-1} + A_0 + A_1.$$

We immediately obtain the following facts.

Proposition 2.1. *If Q is irreducible, then P is irreducible, and if P is irreducible, both A_{-1} and A_1 are nonzero and A_* is irreducible.*

Instead of assuming Q being irreducible, we assume only the following condition throughout this section.

Assumption 2.1. *Both A_{-1} and A_1 are nonzero matrices.*

We also assume the following condition throughout this section.

Assumption 2.2. *All iterates of A_* are finite, i.e., for any $n \in \mathbb{Z}_+$, $A_*^n < \infty$.*

Remark 2.1. Under Assumption 2.2, all multiple products of A_{-1} , A_0 and A_1 becomes finite, i.e., for any $n \in \mathbb{N}$ and for any $\mathbf{i}_{(n)} = (i_1, i_2, \dots, i_n) \in \mathbb{H}^n$, $A_{i_1} A_{i_2} \times \dots \times A_{i_n} < \infty$. Hence, we can define matrices corresponding to the rate matrix and G-matrix of a quasi-birth-and-death process (QBD process for short). A sufficient condition on which Assumption 2.2 holds is that the sum of every row of A_* is bounded, i.e., there exists some positive real number c such that $A_* \mathbf{e} \leq c \mathbf{e}$; in that case, matrix $c^{-1} A_*$ becomes substochastic.

We define a rate matrix R and G-matrix G with respect to the triplet (A_{-1}, A_0, A_1) . Since A_* is not assumed to be stochastic or substochastic, we cannot use probabilistic arguments for the purpose. However, it should be noted that the following definitions are essentially the same as those used in the case where A_* is stochastic or substochastic. Define the following sets of index sequences: for $n \geq 1$ and for $m \geq 1$,

$$\begin{aligned}\mathcal{I}_n &= \left\{ \mathbf{i}_{(n)} \in \mathbb{H}^n : \sum_{l=1}^k i_l \geq 0 \text{ for } k \in \mathbb{N}_{n-1} \text{ and } \sum_{l=1}^n i_l = 0 \right\}, \\ \mathcal{I}_{D,m,n} &= \left\{ \mathbf{i}_{(n)} \in \mathbb{H}^n : \sum_{l=1}^k i_l \geq -m + 1 \text{ for } k \in \mathbb{N}_{n-1} \text{ and } \sum_{l=1}^n i_l = -m \right\}, \\ \mathcal{I}_{U,m,n} &= \left\{ \mathbf{i}_{(n)} \in \mathbb{H}^n : \sum_{l=1}^k i_l \geq 1 \text{ for } k \in \mathbb{N}_{n-1} \text{ and } \sum_{l=1}^n i_l = m \right\},\end{aligned}$$

where $\mathbf{i}_{(n)} = (i_1, i_2, \dots, i_n)$. For $n \geq 1$, let $Q_{0,0}^{(n)}$, $D^{(n)}$ and $U^{(n)}$ be defined as

$$\begin{aligned}Q_{0,0}^{(n)} &= \sum_{\mathbf{i}_{(n)} \in \mathcal{I}_n} A_{i_1} A_{i_2} \cdots A_{i_n}, \\ D^{(n)} &= \sum_{\mathbf{i}_{(n)} \in \mathcal{I}_{D,1,n}} A_{i_1} A_{i_2} \cdots A_{i_n}, \\ U^{(n)} &= \sum_{\mathbf{i}_{(n)} \in \mathcal{I}_{U,1,n}} A_{i_1} A_{i_2} \cdots A_{i_n}.\end{aligned}$$

Under Assumption 2.2, $Q_{0,0}^{(n)}$, $D^{(n)}$ and $U^{(n)}$ are finite for every $n \geq 1$. Let N , R and G be defined as

$$N = \sum_{n=0}^{\infty} Q_{0,0}^{(n)}, \quad G = \sum_{n=1}^{\infty} D^{(n)}, \quad R = \sum_{n=1}^{\infty} U^{(n)},$$

where $Q_{0,0}^{(0)} = I$. Then, the following properties hold.

Proposition 2.2. (i) R and G satisfy the following equations, including the case where both the sides of the equations diverge.

$$R = A_1 N, \quad G = N A_{-1}. \tag{2.1}$$

(ii) R and G satisfy the following equations, including the case where both the sides of the equations diverge.

$$R = R^2 A_{-1} + R A_0 + A_1, \tag{2.2}$$

$$G = A_{-1} + A_0 G + A_1 G^2. \tag{2.3}$$

(iii) N satisfies the following equation, including the case where both the sides of the equation diverge.

$$N = I + (A_0 + A_1 G)N. \quad (2.4)$$

Since the proof for Proposition 2.2 is elementary, we give it on Appendix B. Equation (2.4) will be used in the form of $(I - H)N = I$, where $H = A_0 + A_1 G$. Consider the following matrix quadratic equations of X :

$$X = X^2 A_{-1} + X A_0 + A_1, \quad (2.5)$$

$$X = A_{-1} + A_0 X + A_1 X^2. \quad (2.6)$$

By Proposition 2.2, R and G are solutions to equations (2.5) and (2.6), respectively. Consider the following sequences of matrices:

$$X_0^{(1)} = O, \quad X_n^{(1)} = (X_{n-1}^{(1)})^2 A_{-1} + X_{n-1}^{(1)} A_0 + A_1, \quad n \geq 1, \quad (2.7)$$

$$X_0^{(2)} = O, \quad X_n^{(2)} = A_{-1} + A_0 X_{n-1}^{(2)} + A_1 (X_{n-1}^{(2)})^2, \quad n \geq 1. \quad (2.8)$$

Like the case of usual QBD process, we can demonstrate that both the sequences $\{X_n^{(1)}\}_{n \geq 0}$ and $\{X_n^{(2)}\}_{n \geq 0}$ are nondecreasing and that if a nonnegative solution X^* to equation (2.5) (equation (2.6)) exists, then for any $n \geq 0$, $X^* \geq X_n^{(1)}$ (resp. $X^* \geq X_n^{(2)}$). Furthermore, letting R_n and G_n be defined as

$$R_n = \sum_{k=1}^n U^{(k)}, \quad G_n = \sum_{k=1}^n D^{(k)}$$

we can also demonstrate that, for any $n \geq 1$, $R_n \leq X_n^{(1)}$ and $G_n \leq X_n^{(2)}$ hold. Hence, we obtain the following facts.

Lemma 2.1. *R exists if and only if the minimum nonnegative solution to equation (2.5) exists, and the former is identical to the latter. Similarly, G exists if and only if the minimum nonnegative solution to equation (2.6) exists, and the former is identical to the latter.*

Remark 2.2. *The minimal nonnegative solutions to equations (2.5) and (2.6) are given by $X_\infty^{(1)} = \lim_{n \rightarrow \infty} X_n^{(1)}$ and $X_\infty^{(2)} = \lim_{n \rightarrow \infty} X_n^{(2)}$, respectively, if they exist.*

Since the proof for Lemma 2.1 is straightforward, we omit it. For $z > 0$, let a matrix $A_*(z)$ be defined as

$$A_*(z) = z^{-1} A_{-1} + A_0 + z A_1,$$

where $A_* = A_*(1)$. Note that if A_* is irreducible, then $A_*(z)$ is also irreducible for any $z > 0$. $A_*(z)$ satisfies the following equations called Winer-Hopf factorizations (e.g., see Lemma 3.1 of Miyazawa and Zhao [8]).

Lemma 2.2. *Let z be a positive number. If both R and G exist, then we have*

$$I - A_*(z) = (I - zR)(I - H)(I - z^{-1}G). \quad (2.9)$$

where $H = A_0 + A_1 N A_{-1}$.

Proof. If R and G exists, then they are finite and $(I + zR)(I + H)(I + z^{-1}G)$ is also finite. Hence, the right hand side of equation (2.9) exists and, by some manipulation, we can see that it equals the left hand side. \square

Let $\chi(z)$ be the reciprocal of the convergence parameter of $A_*(z)$, i.e., $\chi(z) = \text{cp}(A_*(z))^{-1}$. According to Kingman [5], we say that a positive function $f(x)$ is superconvex in x if $\log f(x)$ is convex in x ; a superconvex function is also a convex function. $\chi(z)$ satisfies the following properties.

Lemma 2.3. *Assume P is irreducible. Then, $\chi(e^s)$ is superconvex in $s \in \mathbb{R}$, and it is unbounded in both directions, i.e., $\lim_{s \rightarrow -\infty} \chi(e^s) = \lim_{s \rightarrow \infty} \chi(e^s) = \infty$.*

Proof. Note that, since P is irreducible, A_* is also irreducible. Furthermore, every element of $A_*(e^s)$ is superconvex in s . Hence, by Lemma A.1 in Appendix A, $\chi(e^s)$ is also superconvex in s . For $n \in \mathbb{N}$, $j \in \mathbb{Z}_+$ and $s \in \mathbb{R}$, $A_*(e^s)^n$ satisfies

$$[A_*(e^s)^n]_{jj} = \sum_{\mathbf{i}_n \in \mathbb{H}^n} [A_{i_1} A_{i_2} \times \cdots \times A_{i_n}]_{jj} e^{s \sum_{k=1}^n i_k}, \quad (2.10)$$

where $\mathbf{i}_n = (i_1, i_2, \dots, i_n)$. Since P is irreducible, there exist $n_0 > 1$ and $\mathbf{i}_{n_0} \in \mathbb{H}^{n_0}$ such that $[A_{i_1} A_{i_2} \times \cdots \times A_{i_{n_0}}]_{jj} > 0$ and $\sum_{k=1}^{n_0} i_k = 1$. For such a n_0 , we have $[A_*(e^s)^{n_0}]_{jj} \geq c e^s$ for some $c > 0$. This implies that, for any $m \in \mathbb{N}$, $[A_*(e^s)^{n_0 m}]_{jj} \geq c^m e^{s m}$ and we have

$$\chi(e^s) = \limsup_{m \rightarrow \infty} ([A_*(e^s)^m]_{jj})^{\frac{1}{m}} \geq \limsup_{m \rightarrow \infty} ([A_*(e^s)^{n_0 m}]_{jj})^{\frac{1}{n_0 m}} \geq c^{\frac{1}{n_0}} e^{\frac{s}{n_0}}.$$

Therefore, $\lim_{s \rightarrow \infty} \chi(e^s) = \infty$. Analogously, we can obtain that $\chi(e^s) \geq c^{\frac{1}{n_0}} e^{-\frac{s}{n_0}}$ for some $n_0 \in \mathbb{N}$ and $c > 0$, and this implies that $\lim_{s \rightarrow -\infty} \chi(e^s) = \infty$. \square

Let γ^* be the infimum of $\chi(z)$, i.e.,

$$\gamma^* = \inf_{z > 0} \chi(z).$$

By Lemma 2.3, if P is irreducible, there exists a $z > 0$ such that $\gamma^* = \chi(z)$. In our case, the following properties corresponding to those in Lemma 2.3 of Kijima [4] hold.

Lemma 2.4. *Assume A_* is irreducible.*

- (i) *If $\gamma^* \leq 1$, then the nonnegative solutions to matrix quadratic equations (2.5) and (2.6) exist.*
- (ii) *If the nonnegative solution to matrix quadratic equation (2.5), denoted by X^* , exists and there exist a positive number γ_0 and nonnegative nonzero vector \mathbf{u} such that $\mathbf{u}^\top X^* = \gamma_0 \mathbf{u}^\top$, then $\gamma^* \leq 1$.*
- (ii') *If the nonnegative solution to matrix quadratic equation (2.6), denoted by X^* , exists and there exist a positive number γ_0 and nonnegative nonzero vector \mathbf{v} such that $X^* \mathbf{v} = \gamma_0 \mathbf{v}$, then $\gamma^* \leq 1$.*

Proof. *Statement (i)* Assume $\gamma^* \leq 1$ and let z^* be a positive number satisfying $\chi(z^*) = \gamma^*$. Since $A_*(z^*)$ is irreducible, by Lemma 1 and Theorem 1 of Pruitt [16], there exists a positive vector \mathbf{u} satisfying $\mathbf{u}^\top A_*(z^*) \leq \gamma^* \mathbf{u}^\top$. For this \mathbf{u} , we obtain, by induction using formula (2.7), inequality

$\mathbf{u}^\top X_n^{(1)} \leq (z^*)^{-1} \mathbf{u}^\top$ for any $n \geq 0$. Hence, the sequence $\{X_n^{(n)}\}$ is element-wise nondecreasing and bounded, and the limit of the sequence, which is the minimum nonnegative solution to equation (2.5), exists. Existence of the minimum nonnegative solution to equation (2.6) is analogously proved.

Statements (ii) and (ii') Assume the condition of Statement (ii). Then, we have

$$\gamma_0 \mathbf{u}^\top = \mathbf{u}^\top X^* = \mathbf{u}^\top ((X^*)^2 A_{-1} + X^* A_0 + A_1) = \gamma_0 \mathbf{u}^\top A_*(\gamma_0^{-1}), \quad (2.11)$$

and this leads us to $\gamma^* \leq \chi(\gamma_0^{-1}) = \text{cp}(A_*(\gamma_0^{-1}))^{-1} \leq 1$. Statement (ii') can analogously be proved. \square

Let a set $\bar{\Gamma}$ be defined as

$$\bar{\Gamma} = \{s \in \mathbb{R}; \chi(e^s) \leq 1\}.$$

By Lemma 2.3, if P is irreducible and $\gamma^* < 1$, $\bar{\Gamma}$ is a line segment and there exist just two real solutions to equation $\chi(z) = 1$; we denote the solutions by $\underline{\zeta}$ and $\bar{\zeta}$, where $\underline{\zeta} < \bar{\zeta}$. When $\gamma^* = 1$, we define $\underline{\zeta}$ and $\bar{\zeta}$ as $\underline{\zeta} = \min\{z > 0; \chi(z) = 1\}$ and $\bar{\zeta} = \max\{z > 0; \chi(z) = 1\}$, respectively. For the convergence parameters of R , G , we have the following lemma, which corresponds to Lemma 2.2 of He et al. [3] and Lemma 2.3 of Miyazawa [9].

Lemma 2.5. *Assume P , R and G are irreducible. If $\gamma^* \leq 1$, then we have*

$$\text{cp}(R) = \bar{\zeta}, \quad \text{cp}(G) = \underline{\zeta}^{-1}. \quad (2.12)$$

Proof. Let \mathbf{u} be a positive vector satisfying $\mathbf{u}^\top A_*(z) \leq \mathbf{u}^\top$; such \mathbf{u} exists if $\chi(z) \leq 1$. As mentioned in the proof of Lemma 2.4, for $X_n^{(1)}$ defined by formula (2.7), if $\chi(z) \leq 1$, then we have $\mathbf{u}^\top X_n^{(1)} \leq z^{-1} \mathbf{u}^\top$ for any $n \geq 0$ and $\mathbf{u}^\top R \leq z^{-1} \mathbf{u}^\top$. Analogously, if $\chi(z) \leq 1$, then there exists a positive vector \mathbf{v} satisfying $A_*(z) \mathbf{v} \leq \mathbf{v}$ and we have $G \mathbf{v} \leq z \mathbf{v}$. Hence, we obtain that, for some positive vector \mathbf{u} and \mathbf{v} , $\mathbf{u}^\top R \leq \bar{\zeta}^{-1} \mathbf{u}^\top$ and $G \mathbf{v} \leq \underline{\zeta} \mathbf{v}$, and this leads us to $\text{cp}(R) \geq \bar{\zeta}$ and $\text{cp}(G) \geq \underline{\zeta}^{-1}$.

Next, we prove that $\text{cp}(R) > \bar{\zeta}$ is impossible. Suppose $\text{cp}(R) = \bar{\zeta} + \varepsilon$ for some $\varepsilon > 0$ and let \mathbf{v} be a positive vector satisfying $R \mathbf{v} \leq (\bar{\zeta} + \varepsilon)^{-1} \mathbf{v}$; such a positive vector exists since R is irreducible. Then, from equations (2.9) and (2.4), we obtain

$$(I - A_*(\bar{\zeta} + \varepsilon))(I - (\bar{\zeta} + \varepsilon)^{-1} G)^{-1} N \mathbf{v} = (I - (\bar{\zeta} + \varepsilon) R) \mathbf{v} \geq \mathbf{0}, \quad (2.13)$$

where $\mathbf{y} = (I - (\bar{\zeta} + \varepsilon)^{-1} G)^{-1} N \mathbf{v}$ is a positive vector. Hence, we obtain $A_*(\bar{\zeta} + \varepsilon) \mathbf{y} \leq \mathbf{y}$. This contradicts that $\text{cp}(A_*(\bar{\zeta} + \varepsilon)) = \chi(\bar{\zeta} + \varepsilon)^{-1} < 1$, and we obtain $\text{cp}(R) = \bar{\zeta}$. In a similar manner, we also see that $\text{cp}(G) > \underline{\zeta}^{-1}$ is impossible, and this completes the proof. \square

Lemma 2.6. *Assume P is irreducible. If $\gamma^* \leq 1$, then we have*

$$\text{cp}(H) \geq (\gamma^*)^{-1}. \quad (2.14)$$

Proof. Let z^* is a positive number satisfying $\chi(z^*) = \gamma^*$ and \mathbf{u}^* a positive vector satisfying $(\mathbf{u}^*)^\top A_*(z^*) \leq \gamma^* (\mathbf{u}^*)^\top$. Since $\chi(z^*) = \gamma^* \leq 1$, we have $(\mathbf{u}^*)^\top R \leq (z^*)^{-1} (\mathbf{u}^*)^\top$. Hence,

$$(\mathbf{u}^*)^\top H = (\mathbf{u}^*)^\top (R A_{-1} + A_0) \leq (\mathbf{u}^*)^\top ((z^*)^{-1} A_{-1} + A_0) \leq (\mathbf{u}^*)^\top A_*(z^*) \leq \gamma^* (\mathbf{u}^*)^\top.$$

This implies that $\text{cp}(H) \geq (\gamma^*)^{-1}$. \square

With respect to $\bar{\zeta}$ -invariant measure of R and $\underline{\zeta}^{-1}$ -invariant vector of G , we obtain the following properties.

Lemma 2.7. *Assume that P , R and G are irreducible and $\gamma^* < 1$. For a positive vector \mathbf{u} , $\mathbf{u}^\top A_*(\bar{\zeta}) = \mathbf{u}^\top$ if and only if $\mathbf{u}^\top R = \bar{\zeta}^{-1} \mathbf{u}^\top$. Furthermore, for a positive vector \mathbf{v} , $A_*(\underline{\zeta})\mathbf{v} = \mathbf{v}$ if and only if $G\mathbf{v} = \underline{\zeta}\mathbf{v}$.*

Proof. For $\mathbf{u} > 0$, if $\mathbf{u}^\top R = \bar{\zeta}^{-1} \mathbf{u}^\top$, then we immediately obtain, by Lemma 2.5, $\mathbf{u}^\top A_*(\bar{\zeta}) = \mathbf{u}^\top$. Analogously, we see that, for $\mathbf{v} > 0$, if $G\mathbf{v} = \underline{\zeta}\mathbf{v}$, then $A_*(\underline{\zeta})\mathbf{v} = \mathbf{v}$.

Next, we consider the inverse. Since $\gamma^* < 1$, $\underline{\zeta} < \bar{\zeta}$ and we obtain, from equations (2.9) and (2.4),

$$(I - A_*(\bar{\zeta}))(I - (\bar{\zeta})^{-1}G)^{-1}N = (I - \bar{\zeta}R), \quad (2.15)$$

$$N(I - \underline{\zeta}R)^{-1}(I - A_*(\underline{\zeta})) = (I - (\underline{\zeta})^{-1}G). \quad (2.16)$$

From these equations, we see that, for $\mathbf{u} > 0$, if $\mathbf{u}^\top A_*(\bar{\zeta}) = \mathbf{u}^\top$, then $\mathbf{u}^\top R = \bar{\zeta}^{-1} \mathbf{u}^\top$ and that, for $\mathbf{v} > 0$, if $A_*(\underline{\zeta})\mathbf{v} = \mathbf{v}$, then $G\mathbf{v} = \underline{\zeta}\mathbf{v}$. \square

Remark 2.3. *If $\gamma^* = 1$, it may hold that $\underline{\zeta} = \bar{\zeta}$. In such a case, equation (2.15) holds only if G is $\underline{\zeta}$ -transient, and equation (2.16) holds only if R is $\bar{\zeta}$ -transient. By Theorem 1 of Pruitt [16], if R is $\bar{\zeta}$ -recurrent, then there exists a unique positive vector \mathbf{u} , up to positive multiples, satisfying $\mathbf{u}^\top R = \bar{\zeta}^{-1} \mathbf{u}^\top$ and, by equation (2.9), this \mathbf{u} satisfies $\mathbf{u}^\top A_*(\bar{\zeta}) = \mathbf{u}^\top$. A similar thing also holds for G .*

Remark 2.4. *By Theorem 2 of Pruitt [16], if the number of nonzero elements of every row of A_* is finite, there exists a positive vector \mathbf{u} satisfying $\mathbf{u}^\top A_*(\bar{\zeta}) = \mathbf{u}^\top$. Also, if the number of nonzero elements of every column of A_* is finite, there exists a positive vector \mathbf{v} satisfying $A_*(\underline{\zeta})\mathbf{v} = \mathbf{v}$.*

Let \tilde{Q} be the fundamental matrix of Q , i.e., $\tilde{Q} = \sum_{n=0}^{\infty} Q^n$. For $n \geq 0$, $Q_{0,0}^{(n)}$ is the $(0,0)$ -block of Q^n , and N is that of \tilde{Q} . We have the following facts.

Lemma 2.8. *Assume Q is irreducible.*

- (i) \tilde{Q} is finite if R or G exists.
- (ii) If the number of nonzero elements of any row of A_1 is finite and \tilde{Q} is finite, then R exists.
- (iii) If the number of nonzero elements of any column of A_{-1} is finite and \tilde{Q} is finite, then G exists.

Proof. Since Q is irreducible, by Theorem 6.1 of Seneta [17], the radius of convergence of any element of $\sum_{n=0}^{\infty} (zQ)^n$ takes a common value. Hence, \tilde{Q} is finite if and only if N is finite. Furthermore, irreducibility of Q implies that both A_1 and A_{-1} are nonzero. Hence, from expression (2.1), we obtain statements (i) to (iii). \square

Recall that γ^* is defined as $\gamma^* = \inf_{z>0} \text{cp}(A_*(z))^{-1}$. From Lemmas 2.1, 2.4 and 2.8, we immediately obtain the following proposition.

Proposition 2.3. *Assume Q is irreducible. If $\gamma^* \leq 1$, then $\text{cp}(Q) \geq 1$.*

For the convergence parameter of Q , we obtain, by this proposition, the following fact.

Lemma 2.9. *Assume Q is irreducible. Then, we have $\text{cp}(Q) \geq (\gamma^*)^{-1}$.*

Proof. For $\beta > 0$, βQ is a nonnegative block tri-diagonal matrix, whose block matrices are given by βA_{-1} , βA_0 and βA_1 . Let $\gamma^*(\beta)$ be defined as

$$\gamma^*(\beta) = \inf_{z>0} \text{cp}(\beta A_*(z))^{-1} = \beta \inf_{z>0} \text{cp}(A_*(z))^{-1} = \beta \gamma^*. \quad (2.17)$$

Then, by Proposition 2.3, if $\gamma^*(\beta) = \beta \gamma^* \leq 1$, $\text{cp}(\beta Q) \geq 1$. Since $\text{cp}(\beta Q) = \beta^{-1} \text{cp}(Q)$, this implies that if $\beta \leq (\gamma^*)^{-1}$, then $\text{cp}(Q) \geq \beta$. Therefore, setting β at $(\gamma^*)^{-1}$, we obtain $\text{cp}(Q) \geq (\gamma^*)^{-1}$. \square

3 3-D MMRW and asymptotic properties of its fundamental matrix

3.1 Model description

Consider a discrete-time three-dimensional process $\{\mathbf{X}_n\} = \{(X_{1,n}, X_{2,n}, X_{3,n})\}$ on \mathbb{Z}^3 and a background process $\{J_n\}$ on a finite set $S_0 = \{1, 2, \dots, s_0\}$, where s_0 is the number of elements in S_0 . We assume that individual processes $\{X_{1,n}\}$ to $\{X_{3,n}\}$ are skip free, which means that their increments take only values in \mathbb{H} . Furthermore, we assume that the joint process $\{\mathbf{Y}_n\} = \{(\mathbf{X}_n, J_n)\}$ is Markovian and that the transition probabilities of the three-dimensional process $\{\mathbf{X}_n\}$ are modulated depending on the state of the background process $\{J_n\}$. This modulation is assumed to be space homogeneous. We refer to this process as a discrete-time three-dimensional skip-free Markov modulate random walk (3D-MMRW for short). The state space of the 3D-MMRW is given by $\mathbb{S} = \mathbb{Z}^3 \times S_0$. According to QBD processes, we call the process $\{\mathbf{X}_n\}$ the level process and the background process $\{J_n\}$ the phase process.

Let $P = (p_{\mathbf{y}, \mathbf{y}'}, \mathbf{y}, \mathbf{y}' \in \mathbb{S})$, where $\mathbf{y} = (\mathbf{x}, j) = (x_1, x_2, x_3, j)$ and $\mathbf{y}' = (\mathbf{x}', j') = (x'_1, x'_2, x'_3, j')$, be the transition matrix of the 3D-MMRW $\{\mathbf{Y}_n\}$, i.e., $p_{\mathbf{y}, \mathbf{y}'} = \mathbb{P}(\mathbf{Y}_{n+1} = \mathbf{y}' | \mathbf{Y}_n = \mathbf{y})$. By the property of skip-free, each element of P , say $p_{(\mathbf{x}, j), (\mathbf{x}', j')}$, is nonzero only if $\mathbf{x}' - \mathbf{x} \in \mathbb{H}^3$; by the property of space-homogeneity, for any $\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{Z}^3$, $p_{(\mathbf{x}+\mathbf{k}, j), (\mathbf{x}'+\mathbf{k}, j')} = p_{(\mathbf{x}, j), (\mathbf{x}', j')}$. Thus, the transition matrix P can be described only by the following 27 block matrices of dimension $s_0 \times s_0$:

$$A_{\mathbf{k}} = A_{k_1, k_2, k_3} = (p_{(0,0,0,j)(k_1, k_2, k_3, j')}, j, j' \in S_0), \mathbf{k} \in \mathbb{H}^3,$$

i.e., for $\mathbf{x}, \mathbf{x}' \in \mathbb{Z}^3$

$$(p_{(\mathbf{x}, j)(\mathbf{x}', j')}, j, j' \in S_0) = \begin{cases} A_{\mathbf{x}' - \mathbf{x}}, & \text{if } \mathbf{x}' - \mathbf{x} \in \mathbb{H}^3, \\ O, & \text{otherwise.} \end{cases} \quad (3.1)$$

For $k_2, k_3 \in \mathbb{H}$ and for $z_1, z_2, z_3 > 0$, let matrix functions $A_{*, k_2, k_3}(z_1)$, $A_{**, k_3}(z_1, z_2)$ and $A_{***}(z_1, z_2, z_3)$ be defined as

$$\begin{aligned} A_{*, k_2, k_3}(z_1) &= \sum_{k_1 \in \mathbb{H}} z_1^{k_1} A_{k_1, k_2, k_3}, & A_{**, k_3}(z_1, z_2) &= \sum_{k_1, k_2 \in \mathbb{H}} z_1^{k_1} z_2^{k_2} A_{k_1, k_2, k_3}, \\ A_{***}(z_1, z_2, z_3) &= \sum_{k_1, k_2, k_3 \in \mathbb{H}} z_1^{k_1} z_2^{k_2} z_3^{k_3} A_{k_1, k_2, k_3}. \end{aligned}$$

Other matrix functions such as $A_{*,k_2,*}(z_1, z_3)$ are analogously defined. For $k_1, k_2, k_3 \in \mathbb{H} \cup \{*\}$, when all the variables of $A_{k_1,k_2,k_3}(\cdot)$ are set at unity, we denote it by A_{k_1,k_2,k_3} , for example, $A_{*,-1,*} = A_{*,-1,*}(1, 1)$. A_{***} is the transition probability matrix of the background process $\{J_n\}$, and if it is irreducible, then it is positive recurrent since its dimension is finite. We denote by π_{***} the stationary probability distribution of A_{***} when it exists. The mean drift vector of the process $\{Y_n\}$, $\mathbf{a} = (a_1, a_2, a_3)$, is given as

$$\begin{aligned} a_1 &= \pi_{***}(-A_{-1,**} + A_{1,**})\mathbf{e}, & a_2 &= \pi_{***}(-A_{*,-1,*} + A_{*,1,*})\mathbf{e}, \\ a_3 &= \pi_{***}(-A_{**, -1} + A_{**, 1})\mathbf{e}. \end{aligned} \quad (3.2)$$

Let a truncated matrix Q be defined as $Q = (p_{\mathbf{y}, \mathbf{y}'}, \mathbf{y}, \mathbf{y}' \in \mathbb{S}_+)$, where $\mathbb{S}_+ = \mathbb{Z}_+^3 \times S_0$. This Q governs transitions of the 3D-MMRW $\{Y_n\}$ on the nonnegative subspace of \mathbb{Z}^3 . Our main subject is the fundamental matrix of Q given by

$$\tilde{Q} = (\tilde{q}_{\mathbf{y}, \mathbf{y}'}, \mathbf{y}, \mathbf{y}' \in \mathbb{S}_+) = \sum_{n=0}^{\infty} Q^n, \quad (3.3)$$

where $Q^n = (q_{\mathbf{y}, \mathbf{y}'}^{(n)})$ is recursively defined as $q_{\mathbf{y}, \mathbf{y}'}^{(0)} = \delta_{\mathbf{y}, \mathbf{y}'}$ and $q_{\mathbf{y}, \mathbf{y}'}^{(n)} = \sum_{\mathbf{y}'' \in \mathbb{S}_+} q_{\mathbf{y}, \mathbf{y}''}^{(n-1)} q_{\mathbf{y}'', \mathbf{y}'}$ for $n \geq 1$; $\delta_{\mathbf{y}, \mathbf{y}'}$ is the delta function that takes unity if $\mathbf{y} = \mathbf{y}'$, otherwise zero. Each element of \tilde{Q} , say $\tilde{q}_{\mathbf{y}, \mathbf{y}'}$, is the expected number of visits to state \mathbf{y}' until the process $\{Y_n\}$ starting from state $\mathbf{y} \in \mathbb{S}_+$ leaves the subspace \mathbb{S}_+ for the first time, i.e.,

$$\tilde{q}_{\mathbf{y}, \mathbf{y}'} = \mathbb{E} \left(\sum_{n=0}^{\sigma_+-1} 1(Y_n = \mathbf{y}') \mid Y_0 = \mathbf{y} \right), \quad \mathbf{y}, \mathbf{y}' \in \mathbb{S}_+, \quad (3.4)$$

where $1(\cdot)$ is an indicator function and σ_+ is a stopping time defined as

$$\sigma_+ = \inf\{n \geq 0; Y_n \in \mathbb{S} \setminus \mathbb{S}_+\}.$$

Therefore, each row of \tilde{Q} is also called an occupation measure. A sufficient condition on which \tilde{Q} exists (is finite) is given as follows.

Theorem 3.1. *Assume that A_{***} is irreducible. If all the elements of the mean drift vector \mathbf{a} given by expression (3.2) are nonzero and at least one element of \mathbf{a} is negative, then every element of \tilde{Q} is finite and, furthermore, the sum of any row of \tilde{Q} is also finite, i.e., for any $\mathbf{y} \in \mathbb{S}_+$, $\sum_{\mathbf{y}' \in \mathbb{S}_+} \tilde{q}_{\mathbf{y}, \mathbf{y}'} < \infty$.*

We give a proof for this theorem on Appendix C. Using the results derived in Section 2, we obtain asymptotic properties of sequences $\{\tilde{q}_{\mathbf{y}, (n, x'_2, x'_3, j')}\}_{n \in \mathbb{Z}_+}$, $\{\tilde{q}_{\mathbf{y}, (x'_1, n, x'_3, j')}\}_{n \in \mathbb{Z}_+}$ and $\{\tilde{q}_{\mathbf{y}, (x'_1, x'_2, n, j')}\}_{n \in \mathbb{Z}_+}$ as n tends to infinity, for every $\mathbf{y} \in \mathbb{S}_+$. For the purpose, we assume the following conditions throughout the rest of this section.

Assumption 3.1. *All elements of the mean drift vector \mathbf{a} are nonzero and at least one element of \mathbf{a} is negative.*

Assumption 3.2. *Q is irreducible and aperiodic.*

Under Assumption 3.1, the conclusion of Theorem 3.1 holds. Under Assumption 3.2, both P and A_{***} are irreducible and aperiodic and every element of \tilde{Q} is positive.

3.2 Block tri-diagonal representation of Q

We consider three kinds of block tri-diagonal representation for the process $\{\mathbf{Y}_n\}$: one is that for a process given as $\{\mathbf{Y}_n^{(1)}\} = \{(X_{1,n}, (X_{2,n}, (X_{3,n}, \hat{J}_n)))\}$, where $X_{1,n}$ is the primary level and $(X_{2,n}, (X_{3,n}, \hat{J}_n))$ the primary phase; $X_{2,n}$ is the secondary level and $(X_{3,n}, \hat{J}_n)$ the secondary phase; the other representations are those for $\{\mathbf{Y}_n^{(2)}\} = \{(X_{2,n}, (X_{1,n}, (X_{3,n}, \hat{J}_n)))\}$ and $\{\mathbf{Y}_n^{(3)}\} = \{(X_{3,n}, (X_{1,n}, (X_{2,n}, \hat{J}_n)))\}$. For $i \in \mathbb{N}_3$, let $Q^{(i)}$ be the transition probability matrices of $\{\mathbf{Y}_n^{(i)}\}$, restricted to \mathbb{S}_+ ; $Q^{(1)}$ is given in block form, as follows:

$$Q^{(1)} = \begin{pmatrix} A_0^{(1)} & A_1^{(1)} & & & \\ A_{-1}^{(1)} & A_0^{(1)} & A_1^{(1)} & & \\ & A_{-1}^{(1)} & A_0^{(1)} & A_1^{(1)} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad (3.5)$$

where the block matrices are given by, for $k, l \in \mathbb{H}$,

$$A_k^{(1)} = \begin{pmatrix} A_{k,0}^{(1)} & A_{k,1}^{(1)} & & & \\ A_{k,-1}^{(1)} & A_{k,0}^{(1)} & A_{k,1}^{(1)} & & \\ & A_{k,-1}^{(1)} & A_{k,0}^{(1)} & A_{k,1}^{(1)} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad A_{k,l}^{(1)} = \begin{pmatrix} A_{k,l,0} & A_{k,l,1} & & & \\ A_{k,l,-1} & A_{k,l,0} & A_{k,l,1} & & \\ & A_{k,l,-1} & A_{k,l,0} & A_{k,l,1} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

$Q^{(2)}$ and $Q^{(3)}$ are analogously given. On the subspace \mathbb{S}_+ , each $\{\mathbf{Y}_n^{(i)}\}$ behaves just like a QBD process with countably many phase states, in a stochastic sense, but $Q^{(i)}$ is strictly substochastic. For $i \in \mathbb{N}_3$, let $A_*^{(i)}$ be defined as

$$A_*^{(i)} = A_{-1}^{(i)} + A_0^{(i)} + A_1^{(i)};$$

$A_*^{(i)}$ is the transition probability matrix of the primary phase process of $\{\mathbf{Y}_n^{(i)}\}$ when the primary level is greater than zero. We note that $A_*^{(i)}$ is also strictly substochastic.

According to Section 2, we define, for $i \in \mathbb{N}_3$, the rate matrix of $Q^{(i)}$ and denote it by $R^{(i)}$; as mentioned after, under Assumption 3.1, $R^{(i)}$ always exists. $R^{(i)}$ is the minimal nonnegative solution to the matrix quadratic equation:

$$R^{(i)} = (R^{(i)})^2 A_{-1}^{(i)} + R^{(i)} A_0^{(i)} + A_1^{(i)}. \quad (3.6)$$

For $i \in \mathbb{N}_3$, let $\tilde{Q}^{(i)}$ be the fundamental matrices of $Q^{(i)}$. Under Assumption 3.1, since the sum of any row of \tilde{Q} is finite, $\tilde{Q}^{(i)}$ satisfies the same property. Let $N^{(i)}$ be the $(0,0)$ -block of $\tilde{Q}^{(i)}$, then we have (see Proposition 2.2)

$$R^{(i)} = A_1^{(i)} N^{(i)}. \quad (3.7)$$

From this relation, we obtain the following facts.

Proposition 3.1. *Under Assumption 3.2, for $i \in \mathbb{N}_3$, $R^{(i)}$ is irreducible if and only if every row of $A_1^{(i)}$ has at least one positive element.*

Proof. First, suppose that every row of $A_1^{(i)}$ has at least one positive element. Under Assumption 3.2, every element of $N^{(i)}$ is positive, and hence, every element of $A_1^{(i)}N^{(i)}$ is also positive. This implies that $R^{(i)}$ is a positive matrix and it is irreducible. Next, suppose that there exists a row of $A_1^{(i)}$, say the (x_1, x_2, x_3, j) -row of $A_1^{(i)}$, such that every element of the row is zero. Then, every element of the (x_1, x_2, x_3, j) -row of $A_1^{(i)}N^{(i)}$ is also zero and $R^{(i)}$ cannot be irreducible. As a result, we see that the assertion of the proposition holds. \square

Hereafter, we assume the following condition throughout the rest of this section.

Assumption 3.3. *For $i \in \mathbb{N}_3$, every row of $A_1^{(i)}$ has at least one positive element. This is equivalent to that*

$$\sum_{k,l \in \mathbb{H}_+} A_{1,k,l} \mathbf{e} > \mathbf{0}, \quad \sum_{k,l \in \mathbb{H}_+} A_{k,1,l} \mathbf{e} > \mathbf{0}, \quad \sum_{k,l \in \mathbb{H}_+} A_{k,l,1} \mathbf{e} > \mathbf{0}. \quad (3.8)$$

From the proof of Proposition 3.1, we immediately obtain the following corollary.

Corollary 3.1. *Under Assumptions 3.2 and 3.3, for $i \in \mathbb{N}_3$, $R^{(i)}$ is aperiodic.*

3.3 Convergence parameter of the rate matrices

In this subsection, we obtain, for $i \in \mathbb{N}_3$, a lower bound of the convergence parameter of $R^{(i)}$. Since the lower bounds for $\text{cp}(R^{(1)})$, $\text{cp}(R^{(2)})$ and $\text{cp}(R^{(3)})$ can be derived in the same way, we explain only about $\text{cp}(R^{(1)})$.

For $z_1, z_2, z_3 > 0$, let matrix functions $A_*^{(1)}(z_1)$ and $A_{**}^{(1)}(z_1, z_2)$ be defined as

$$A_*^{(1)}(z_1) = \sum_{k \in \mathbb{H}} z_1^k A_k^{(1)}, \quad A_{**}^{(1)}(z_1, z_2) = \sum_{l \in \mathbb{H}} z_2^l A_{*,l}^{(1)}(z_1),$$

where, for $l \in \mathbb{H}$, $A_{*,l}^{(1)}(z_1) = \sum_{k \in \mathbb{H}} z_1^k A_{k,l}^{(1)}$ and both $A_*^{(1)}(z_1)$ and $A_{**}^{(1)}(z_1, z_2)$ are nonnegative block tri-diagonal matrices. We can analogously define $A_{***}^{(1)}(z_1, z_2, z_3)$ but it is identical to $A_{***}(z_1, z_2, z_3)$. Under Assumption 3.2, $A_*^{(1)}(z_1)$, $A_{**}^{(1)}(z_1, z_2)$ and $A_{***}^{(1)}(z_1, z_2, z_3)$ are irreducible and aperiodic. Denote by $\chi_*^{(1)}(z_1)$, $\chi_{**}^{(1)}(z_1, z_2)$ and $\chi(z_1, z_2, z_3)$ the reciprocals of the convergence parameters of $A_*^{(1)}(z_1)$, $A_{**}^{(1)}(z_1, z_2)$ and $A_{***}^{(1)}(z_1, z_2, z_3)$, respectively, where we use $\chi(z_1, z_2, z_3)$ instead of $\chi_{***}(z_1, z_2, z_3)$ in order to make notation simple. Since $A_{***}(z_1, z_2, z_3)$ is a nonnegative irreducible matrix with a finite dimension, $\chi(z_1, z_2, z_3)$ is its Perron-Frobenius eigenvalue and we have $\chi(z_1, z_2, z_3) = \text{spr}(A_{***}(z_1, z_2, z_3))$. According to Kingman [5], we say that a positive function $f(x_1, x_2, x_3)$ is superconvex in (x_1, x_2, x_3) if $\log f(x_1, x_2, x_3)$ is convex in (x_1, x_2, x_3) ; a superconvex function is also a convex function. By Lemma A.1 in Appendix A, we immediately obtain the following facts.

Proposition 3.2. *Under Assumption 3.2, $\chi_*^{(1)}(e^{s_1})$ is superconvex in $s_1 \in \mathbb{R}$, $\chi_{**}^{(1)}(e^{s_1}, e^{s_2})$ superconvex in $(s_1, s_2) \in \mathbb{R}^2$ and $\chi(e^{s_1}, e^{s_2}, e^{s_3})$ superconvex in $(s_1, s_2, s_3) \in \mathbb{R}^3$.*

Let a set $\bar{\Gamma}$ be defined as

$$\bar{\Gamma} = \{(s_1, s_2, s_3) \in \mathbb{R}^3; \chi(e^{s_1}, e^{s_2}, e^{s_3}) \leq 1\}. \quad (3.9)$$

Since we have $\chi(1, 1, 1) = \chi(e^0, e^0, e^0) = \text{spr}(A_{***}) = 1$, $\bar{\Gamma}$ contains the point of $(0, 0, 0)$ and thus it is not empty. By Proposition 3.2, $\bar{\Gamma}$ is a convex set. Furthermore, in a manner similar to that used in the proof of Lemma 2.3, we see that if P is irreducible and aperiodic, $\chi(e^{s_1}, e^{s_2}, e^{s_3})$ is unbounded in all directions, and hence obtain the following lemma.

Lemma 3.1. *Under Assumption 3.2, $\bar{\Gamma}$ is bounded.*

For $i \in \mathbb{N}_3$, define the upper and lower extreme values of $\bar{\Gamma}$ with respect to s_i , denoted by s_i^{max} and s_i^{min} , as

$$s_i^{max} = \sup_{(s_1, s_2, s_3) \in \bar{\Gamma}} s_i \geq 0, \quad s_i^{min} = \inf_{(s_1, s_2, s_3) \in \bar{\Gamma}} s_i \leq 0.$$

By Lemma 3.1, the extreme values are finite. We define, for $i \in \mathbb{N}_3$, z_i^{max} and z_i^{min} as $z_i^{max} = e^{s_i^{max}}$ and $z_i^{min} = e^{s_i^{min}}$, respectively. For $i \in \mathbb{N}_3$, let $\chi_{z_i}(z_1, z_2, z_3)$ be a partial derivative defined as $\chi_{z_i}(z_1, z_2, z_3) = (\partial/\partial z_i)\chi(z_1, z_2, z_3)$. Let $\mathbf{u}(z_1, z_2, z_3)$ and $\mathbf{v}(z_1, z_2, z_3)$ be the left and right eigenvector of $A_{***}(z_1, z_2, z_3)$ with respect to the Perron-Frobenius eigenvalue $\chi(z_1, z_2, z_3)$, satisfying $\mathbf{u}(z_1, z_2, z_3)^\top \mathbf{v}(z_1, z_2, z_3) = 1$. Then, we have, for $i \in \mathbb{N}_3$,

$$\chi_{z_i}(z_1, z_2, z_3) = \mathbf{u}(z_1, z_2, z_3)^\top \left(\frac{\partial}{\partial z_i} A_{***}(z_1, z_2, z_3) \right) \mathbf{v}(z_1, z_2, z_3). \quad (3.10)$$

$A_{***}(1, 1, 1)$ is an irreducible stochastic matrix and we have $\mathbf{u}(1, 1, 1)^\top = \boldsymbol{\pi}_{***}$ and $\mathbf{v}(1, 1, 1) = \mathbf{e}$. Hence, for $i \in \mathbb{N}_3$,

$$\chi_{z_i}(1, 1, 1) = \boldsymbol{\pi}_{***} \frac{\partial}{\partial z_i} A_{***}(z_1, z_2, z_3) \Big|_{(z_1, z_2, z_3) = (1, 1, 1)} \mathbf{e} = a_i,$$

where $\mathbf{a} = (a_2, a_2, a_3)$ is the mean drift vector given by expression (3.2). Since $(\partial/\partial s_i)\chi(e^{s_1}, e^{s_2}, e^{s_3}) = e^{s_i} \chi_{z_i}(e^{s_1}, e^{s_2}, e^{s_3})$ for $i \in \mathbb{N}_3$, \mathbf{a} is the normal vector of the tangent plane that contacts $\bar{\Gamma}$ at point $(s_1, s_2, s_3) = (0, 0, 0)$. From this fact, we see that, under Assumption 3.1, $s_i^{max} > 0$ for every $i \in \mathbb{N}_3$ and that $z_i^{max} > 1$ for every $i \in \mathbb{N}_3$. Let γ^* be defined as $\gamma^* = \inf_{z_1, z_2, z_3 > 0} \chi(z_1, z_2, z_3)$. By the results mentioned above, we see $\gamma^* < 1$.

Let $\gamma_*^{(1)}$ be defined as $\gamma_*^{(1)} = \inf_{z_1 > 0} \chi_*^{(1)}(z_1) = \inf_{z_1 > 0} \text{cp}(A_*^{(1)}(z_1))^{-1}$. By Lemma 2.9, we have

$$\text{cp}(A_*^{(1)}(z_1))^{-1} \leq \inf_{z_2 > 0} \text{cp}(A_{**}^{(1)}(z_1, z_2))^{-1}, \quad \text{cp}(A_{**}^{(1)}(z_1, z_2))^{-1} \leq \inf_{z_3 > 0} \chi(z_1, z_2, z_3), \quad (3.11)$$

and this implies that

$$\gamma_*^{(1)} \leq \inf_{z_1, z_2, z_3 > 0} \chi(z_1, z_2, z_3) = \gamma^* < 1. \quad (3.12)$$

Hence, equation $\text{cp}(A_*^{(1)}(z_1)) = 1$ has just two real solutions $z_1 = \underline{\zeta}_*^{(1)}$ and $z_1 = \bar{\zeta}_*^{(1)}$, where $\underline{\zeta}_*^{(1)} < \bar{\zeta}_*^{(1)}$, and by Lemma 2.5, we have $\text{cp}(R^{(1)}) = \bar{\zeta}_*^{(1)}$. Since $\text{cp}(A_*^{(1)}(e^{s_1}))^{-1}$ is convex in $s_1 \in \mathbb{R}$, by expression (3.11), we have

$$\begin{aligned} \bar{\zeta}_*^{(1)} &\geq \sup\{z_1 > 0; \text{cp}(A_*^{(1)}(z_1))^{-1} \leq 1\} \\ &\geq \sup\{z_1 > 0; \inf_{z_2, z_3 > 0} \chi(z_1, z_2, z_3) \leq 1\} = z_1^{max}. \end{aligned} \quad (3.13)$$

For $R^{(2)}$ and $R^{(3)}$, we can obtain the same results. We gather them into the following theorem.

Theorem 3.2. Under Assumptions 3.1 to 3.3, we have, for $i \in \mathbb{N}_3$, $\text{cp}(R^{(i)}) \geq z_i^{\max}$.

Remark 3.1. It is expected that, for $i \in \mathbb{N}_3$, $\text{cp}(R^{(1)}) = z_i^{\max}$; but, due to a weakness of Lemma 2.4, it cannot be proved only by the results obtained in Section 2. Since the dimension of $A_{***}(z_1, z_2, z_3)$ is finite, we see, from Lemma 2.4, that for any $z_1 > 0$ and $z_2 > 0$, “ $\text{cp}(A_{**}(z_1, z_2))^{-1} \leq 1$ ” always implies that $\inf_{z_3 > 0} \chi(z_1, z_2, z_3) = \inf_{z_3 > 0} \text{cp}(A_{***}(z_1, z_2, z_3))^{-1} \leq 1$. However, the dimension of $A_{**}^{(1)}(z_1, z_2)$ is countably infinite and “ $\text{cp}(A_*(z_1))^{-1} \leq 1$ ” does not necessarily imply that $\inf_{z_2 > 0} \text{cp}(A_{**}(z_1, z_2))^{-1} \leq 1$ for any $z_1 > 0$. Hence, we cannot demonstrate that there are no $z_1 > z_1^{\max}$ such that $\text{cp}(A_*(z_1))^{-1} \leq 1$, only by using the results obtained in Section 2.

3.4 Asymptotic properties of \tilde{Q}

For $\mathbf{y} = (\mathbf{x}, j) \in \mathbb{S}_+$ and $\mathbf{x}' \in \mathbb{Z}_+^3$, let a row vector $\tilde{\mathbf{q}}_{\mathbf{y}, \mathbf{x}'}$ be defined as $\tilde{\mathbf{q}}_{\mathbf{y}, \mathbf{x}'} = (\tilde{q}_{\mathbf{y}, (\mathbf{x}', j')}, j' \in S_0)$ and, for $\mathbf{y} = (\mathbf{x}, j) \in \mathbb{S}_+$, $\tilde{\mathbf{q}}_{\mathbf{y}}$ as $\tilde{\mathbf{q}}_{\mathbf{y}} = (\tilde{\mathbf{q}}_{\mathbf{y}, \mathbf{x}'}, \mathbf{x}' \in \mathbb{Z}_+^3) = (\tilde{q}_{\mathbf{y}, \mathbf{y}'}, \mathbf{y}' \in \mathbb{S}_+)$. Furthermore, for $\mathbf{x}, \mathbf{x}' \in \mathbb{Z}_+^3$, let an $|S_0| \times |S_0|$ matrix $N_{\mathbf{x}, \mathbf{x}'}$ be defined as $N_{\mathbf{x}, \mathbf{x}'} = (\tilde{q}_{(\mathbf{x}, j), (\mathbf{x}', j')}, j, j' \in S_0)$ and an $|S_0| \times |\mathbb{Z}_+^3 \times S_0|$ matrix $N_{\mathbf{x}}$ as $N_{\mathbf{x}} = (N_{\mathbf{x}, \mathbf{x}'}, \mathbf{x}' \in \mathbb{Z}_+^3)$, where $\tilde{Q} = (N_{\mathbf{x}, \mathbf{x}'}, \mathbf{x}, \mathbf{x}' \in \mathbb{Z}_+^3)$. Under Assumption 3.1, the sum of any row of \tilde{Q} is finite and we obtain $\tilde{Q}Q < \infty$; this leads us to

$$\tilde{Q} = I + \tilde{Q}Q. \quad (3.14)$$

From this equation, we immediately obtain, for $\mathbf{x} \in \mathbb{Z}_+^3$, the following equation.

$$N_{\mathbf{x}} = (1(\mathbf{x}' = \mathbf{x})I, \mathbf{x}' \in \mathbb{Z}_+^3) + N_{\mathbf{x}}Q. \quad (3.15)$$

For $\mathbf{y} = (\mathbf{x}, j) = (x_1, x_2, x_3, j) \in \mathbb{S}_+$ and for $x'_1, x'_2, x'_3 \in \mathbb{Z}_+$ and $j' \in S_0$, let generating functions $\varphi_{\mathbf{y}, (*, x'_2, x'_3, j')}(z)$, $\varphi_{\mathbf{y}, (x'_1, *, x'_3, j')}(z)$ and $\varphi_{\mathbf{y}, (x'_1, x'_2, *, j')}(z)$ be defined as

$$\begin{aligned} \varphi_{\mathbf{y}, (*, x'_2, x'_3, j')}(z) &= \sum_{n=0}^{\infty} z^n \tilde{q}_{\mathbf{y}, (n, x'_2, x'_3, j')}, & \varphi_{\mathbf{y}, (x'_1, *, x'_3, j')}(z) &= \sum_{n=0}^{\infty} z^n \tilde{q}_{\mathbf{y}, (x'_1, n, x'_3, j')}, \\ \varphi_{\mathbf{y}, (x'_1, x'_2, *, j')}(z) &= \sum_{n=0}^{\infty} z^n \tilde{q}_{\mathbf{y}, (x'_1, x'_2, n, j')}, \end{aligned}$$

and denote their radiuses of convergence by $r_{\mathbf{y}, (*, x'_2, x'_3, j')}$, $r_{\mathbf{y}, (x'_1, *, x'_3, j')}$ and $r_{\mathbf{y}, (x'_1, x'_2, *, j')}$, respectively; for example,

$$r_{\mathbf{y}, (*, x'_2, x'_3, j')} = \sup\{z \geq 0; \varphi_{\mathbf{y}, (*, x'_2, x'_3, j')}(z) < \infty\}.$$

We have the following facts.

Proposition 3.3. Under Assumption 3.2, for any $\mathbf{y}, \mathbf{y}'' \in \mathbb{S}_+$ and for any $x'_1, x'_2, x'_3 \in \mathbb{Z}_+$ and $j' \in S_0$, we have $r_{\mathbf{y}, (*, x'_2, x'_3, j')} = r_{\mathbf{y}'', (*, x'_2, x'_3, j')}$, $r_{\mathbf{y}, (x'_1, *, x'_3, j')} = r_{\mathbf{y}'', (x'_1, *, x'_3, j')}$ and $r_{\mathbf{y}, (x'_1, x'_2, *, j')} = r_{\mathbf{y}'', (x'_1, x'_2, *, j')}$.

Proof. Recall that $\tilde{q}_{\mathbf{y}, (n, x'_2, x'_3, j')}$ is given by

$$\tilde{q}_{\mathbf{y}, (n, x'_2, x'_3, j')} = \mathbb{E} \left(\sum_{m=0}^{\infty} 1(\mathbf{Y}_m = (n, x'_2, x'_3, j')) 1(\sigma_+ > m) \mid \mathbf{Y}_0 = \mathbf{y} \right),$$

where σ_+ is a stopping time defined as $\sigma_+ = \inf\{m \geq 0; \mathbf{Y}_m \in \mathbb{S} \setminus \mathbb{S}_+\}$. Since Q is irreducible, there exists $m_0 \geq 0$ such that $\mathbb{P}(\mathbf{Y}_{m_0} = \mathbf{y}'' | \mathbf{Y}_0 = \mathbf{y}) > 0$. Using this m_0 , we obtain

$$\begin{aligned} & \tilde{q}_{\mathbf{y},(n,x'_2,x'_3,j')} \\ & \geq \mathbb{E} \left(\sum_{m=m_0}^{\infty} 1(\mathbf{Y}_m = (n, x'_2, x'_3, j')) 1(\sigma_+ > m) \middle| \mathbf{Y}_{m_0} = \mathbf{y}'' \right) \mathbb{P}(\mathbf{Y}_{m_0} = \mathbf{y}'' | \mathbf{Y}_0 = \mathbf{y}) \\ & = \tilde{q}_{\mathbf{y}'', (n, x'_2, x'_3, j')} \mathbb{P}(\mathbf{Y}_{m_0} = \mathbf{y}'' | \mathbf{Y}_0 = \mathbf{y}), \end{aligned}$$

and this implies that $r_{\mathbf{y},(*,x'_2,x'_3,j')} \leq r_{\mathbf{y}'',(*,x'_2,x'_3,j')}$. Exchanging \mathbf{y} with \mathbf{y}'' , we also obtain $r_{\mathbf{y}'',(*,x'_2,x'_3,j')} \leq r_{\mathbf{y},(*,x'_2,x'_3,j')}$, and this leads us to $r_{\mathbf{y},(*,x'_2,x'_3,j')} = r_{\mathbf{y}'',(*,x'_2,x'_3,j')}$. The other equations are also obtained in the same manner. \square

Next, we consider matrix generating functions in block form. For $n \in \mathbb{Z}_+$, let matrices $N_{0,n}^{(1)}$ to $N_{0,n}^{(3)}$ be defined as

$$\begin{aligned} N_{0,n}^{(1)} &= (N_{(0,x_2,x_3),(n,x'_2,x'_3)}, (x_2, x_3), (x'_2, x'_3) \in \mathbb{Z}_+^2), \\ N_{0,n}^{(2)} &= (N_{(x_1,0,x_3),(x'_1,n,x'_3)}, (x_1, x_3), (x'_1, x'_3) \in \mathbb{Z}_+^2), \\ N_{0,n}^{(3)} &= (N_{(x_1,x_2,0),(x'_1,x'_2,n)}, (x_1, x_2), (x'_1, x'_2) \in \mathbb{Z}_+^2), \end{aligned}$$

and for $i \in \mathbb{N}_3$, a matrix generating function $\Phi_0^{(i)}(z)$ as

$$\Phi_0^{(i)}(z) = \sum_{n=0}^{\infty} z^n N_{0,n}^{(i)},$$

where, for example, we have

$$\Phi_0^{(1)}(z) = \left((\varphi_{(0,x_2,x_3,j),(*,x'_2,x'_3,j')}(z), j, j' \in S_0), (x_2, x_3), (x'_2, x'_3) \in \mathbb{Z}_+^2 \right).$$

Furthermore, for $i \in \mathbb{N}_3$, let $N_0^{(i)}$ be defined as

$$N_0^{(i)} = (N_{0,n}^{(i)}, n \in \mathbb{Z}_+).$$

For $i \in \mathbb{N}_3$, equation (3.14) is represented as follows:

$$\tilde{Q}^{(i)} = \sum_{n=0}^{\infty} (Q^{(i)})^n = I + \tilde{Q}^{(i)} Q^{(i)}. \quad (3.16)$$

Hence, we have, for $i \in \mathbb{N}_3$,

$$N_0^{(i)} = (I \quad O \quad \cdots) + N_0^{(i)} \tilde{Q}^{(i)}. \quad (3.17)$$

This leads us to, for $i \in \mathbb{N}_3$,

$$\begin{aligned} N_{0,0}^{(i)} &= I + N_{0,0}^{(i)} A_0^{(i)} + N_{0,1}^{(i)} A_{-1}^{(i)}, \\ N_{0,n}^{(i)} &= N_{0,n-1}^{(i)} A_1^{(i)} + N_{0,n}^{(i)} A_0^{(i)} + N_{0,n+1}^{(i)} A_{-1}^{(i)}, \quad n \geq 1, \end{aligned} \quad (3.18)$$

and we have the following solution to equation (3.18):

$$N_{0,n}^{(i)} = N_{0,0}^{(i)}(R^{(i)})^n, \quad N_{0,0}^{(i)} = (I - A_0^{(i)} - R^{(i)}A_{-1}^{(i)})^{-1} = \sum_{k=0}^{\infty} (A_0^{(i)} + R^{(i)}A_{-1}^{(i)})^k, \quad (3.19)$$

where we use the fact that, by Lemma 2.6, $\text{cp}(A_0^{(i)} + R^{(i)}A_{-1}^{(i)}) > 1$. From expression (3.19) and Fubini's theorem, we obtain, for $i \in \mathbb{N}_3$,

$$\Phi_0^{(i)}(z) = \sum_{n=0}^{\infty} z^n N_{0,0}^{(i)}(R^{(i)})^n = N_{0,0}^{(i)} \sum_{n=0}^{\infty} (zR^{(i)})^n. \quad (3.20)$$

From expression (3.20), we obtain the following proposition.

Proposition 3.4. *Under Assumptions 3.1 to 3.3, there exists $(0, x_2, x_3, j) \in \mathbb{S}_+$ such that $r_{(0, x_2, x_3, j), (*, x'_2, x'_3, j')} = \text{cp}(R^{(1)})$ for every $x'_2, x'_3 \in \mathbb{Z}_+$ and every $j' \in S_0$. Also, there exists $(x_1, 0, x_3, j) \in \mathbb{S}_+$ such that $r_{(x_1, 0, x_3, j), (x'_1, *, x'_3, j')} = \text{cp}(R^{(2)})$ for every $x'_1, x'_3 \in \mathbb{Z}_+$ and every $j' \in S_0$; there exists $(x_1, x_2, 0, j) \in \mathbb{S}_+$ such that $r_{(x_1, x_2, 0, j), (x'_1, x'_2, *, j')} = \text{cp}(R^{(3)})$ for every $x'_1, x'_2 \in \mathbb{Z}_+$ and every $j' \in S_0$.*

By Propositions 3.3 and 3.4, we immediately obtain the following theorem.

Proposition 3.5. *Under Assumptions 3.1 to 3.3, for every $\mathbf{y} \in \mathbb{S}_+$ and for every $x'_1, x'_2, x'_3 \in \mathbb{Z}_+$ and every $j' \in S_0$, we have*

$$r_{\mathbf{y}, (*, x'_2, x'_3, j')} = \text{cp}(R^{(1)}), \quad r_{\mathbf{y}, (x'_1, *, x'_3, j')} = \text{cp}(R^{(2)}), \quad r_{\mathbf{y}, (x'_1, x'_2, *, j')} = \text{cp}(R^{(3)}). \quad (3.21)$$

Proof of Proposition 3.4. Let $R^{(1)}(z)$ be defined as $R^{(1)}(z) = \sum_{n=0}^{\infty} (zR^{(1)})^n$. Under Assumptions 3.2 and 3.3, $R^{(1)}$ is irreducible and, by Theorem 6.1 of Seneta [17], the radius of convergence of every element of $R^{(1)}(z)$ is equal to $\text{cp}(R^{(1)})$. Q is irreducible and every element of $N_{0,0}^{(1)}$ is positive. Hence, by equation (3.20), we have

$$\Phi_0^{(1)}(z) < \infty \Rightarrow R^{(1)}(z) < \infty, \quad (3.22)$$

and this implies that $r_{(0, x_2, x_3, j), (*, x'_2, x'_3, j')} \leq \text{cp}(R^{(1)})$ for every $(x_2, x_3), (x'_2, x'_3) \in \mathbb{Z}_+^2$ and every $j, j' \in S_0$. On the other hand, we have $R^{(1)} = A_1^{(1)}N_{0,0}^{(1)}$, and this and Fubini's theorem lead us to

$$zA_1^{(1)}\Phi_0^{(1)}(z) = \sum_{n=0}^{\infty} zA_1^{(1)}N_{0,0}^{(1)}(zR^{(1)})^n = \sum_{n=0}^{\infty} (zR^{(1)})^{n+1} \leq R^{(1)}(z). \quad (3.23)$$

By the definition, the number of positive elements in each row of $A_1^{(1)}$ is finite. Since Q is irreducible, at least one element of $A_1^{(1)}$, say the $((x_2, x_3, j), (x'_2, x'_3, j'))$ -element, is positive. Hence, we have, for every $(x''_2, x''_3) \in \mathbb{Z}_+^2$ and every $j'' \in S_0$,

$$R^{(1)}(z) < \infty \Rightarrow \varphi_{(0, x'_2, x'_3, j'), (*, x''_2, x''_3, j'')}(z) < \infty, \quad (3.24)$$

and this implies that $\text{cp}(R^{(1)}) \leq r_{(0, x'_2, x'_3, j'), (*, x''_2, x''_3, j'')}$. As a result, we obtain that, for every $(x''_2, x''_3) \in \mathbb{Z}_+^2$ and every $j'' \in S_0$, $r_{(0, x'_2, x'_3, j'), (*, x''_2, x''_3, j'')} = \text{cp}(R^{(1)})$.

The other statements of the proposition can be proved in the same manner, and this completes the proof. \square

By Theorem 3.2 and Proposition 3.5, we obtain a main result, as follows.

Theorem 3.3. *Under Assumptions 3.1 to 3.3, for every $\mathbf{y} \in \mathbb{S}_+$ and for every $x'_1, x'_2, x'_3 \in \mathbb{Z}_+$ and every $j' \in S_0$, we have*

$$r_{\mathbf{y},(*,x'_2,x'_3,j')} \geq z_1^{\max}, \quad r_{\mathbf{y},(x'_1,*,x'_3,j')} \geq z_2^{\max}, \quad r_{\mathbf{y},(x'_1,x'_2,*,j')} \geq z_3^{\max}. \quad (3.25)$$

By Theorem 3.3 and the Cauchy-Hadamard theorem, we immediately obtain lower bounds of the directional asymptotic decay rates, as follows.

Corollary 3.2. *Under Assumptions 3.1 to 3.3, for every $\mathbf{y} \in \mathbb{S}_+$ and for every $x'_1, x'_2, x'_3 \in \mathbb{Z}_+$ and every $j' \in S_0$, we have*

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \tilde{q}_{\mathbf{y},(n,x'_2,x'_3,j')} &\leq -s_1^{\max}, & \limsup_{n \rightarrow \infty} \frac{1}{n} \log \tilde{q}_{\mathbf{y},(x'_1,n,x'_3,j')} &\leq -s_2^{\max}, \\ \limsup_{n \rightarrow \infty} \frac{1}{n} \log \tilde{q}_{\mathbf{y},(x'_1,x'_2,n,j')} &\leq -s_3^{\max}. \end{aligned} \quad (3.26)$$

3.5 Three queue model with 1-limited services: An example

Consider a three queue model with 1-limited service, where the queues are labeled by Q_1 , Q_2 and Q_3 . For $i \in \mathbb{N}_3$, class- i customers arrive at Q_i according to a Poisson process with rate λ_i and they receive exponential services with mean $1/\mu_i$. We denote by λ the sum of the arrival rates, i.e., $\lambda = \lambda_1 + \lambda_2 + \lambda_3$. The single server goes around the queues in order Q_1 , Q_2 , Q_3 , without switchover times. The behavior of this model when all the queues are not empty is represented as a continuous-time version of 3D-MMRW, $\{\mathbf{Y}(t)\} = \{(X_1(t), X_2(t), X_3(t), J(t))\}$, where for $i \in \mathbb{N}_3$, $X_i(t)$ is the number of customers in Q_i at time t ; $J(t)$ is the phase state whose state space is given by $S_0 = \{1, 2, 3\}$, where for $i \in \mathbb{N}_3$, “ $J(t) = i$ ” indicates that the server is serving a class- i customer at time t . The infinitesimal generator of $\{\mathbf{Y}(t)\}$, \bar{P} , is represented in block tri-diagonal form and its nonzero block matrices are given as follows:

$$\begin{aligned} \bar{A}_{1,0,0} &= \lambda_1 I, \quad \bar{A}_{0,1,0} = \lambda_2 I, \quad \bar{A}_{0,0,1} = \lambda_3 I, \\ \bar{A}_{-1,0,0} &= \begin{pmatrix} 0 & \mu_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{A}_{0,-1,0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{A}_{0,0,-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mu_3 & 0 & 0 \end{pmatrix}, \\ \bar{A}_{0,0,0} &= \begin{pmatrix} -(\lambda + \mu_1) & 0 & 0 \\ 0 & -(\lambda + \mu_2) & 0 \\ 0 & 0 & -(\lambda + \mu_3) \end{pmatrix}. \end{aligned}$$

By the uniformization technique, we obtain a discrete-time version of $\{\mathbf{Y}(t)\}$, which we denoted by $\{\mathbf{Y}_n\} = \{(X_{1,n}, X_{2,n}, X_{3,n}, J_n)\}$, and the transition probability matrix of $\{\mathbf{Y}_n\}$, P , is given by

$$P = I + \frac{1}{\nu} \bar{P},$$

where we set $\nu = \lambda + \mu_1 + \mu_2 + \mu_3$. Let Q be the truncation of P , considered in the previous subsections, and $\tilde{Q} = (\tilde{q}_{\mathbf{y},\mathbf{y}'})$ the fundamental matrix of Q . In the original three queue mode, $\frac{1}{\nu} \tilde{q}_{(x_1,x_2,x_3,j),(x'_1,x'_2,x'_3,j')}$ is the expected cumulative time in which the process $\{\mathbf{Y}(t)\}$ starting from state $(X_1(0), X_2(0), X_3(0), J(0)) = (x_1 + 1, x_2 + 1, x_3 + 1, j)$ stays state $(X_1(t), X_2(t), X_3(t), J(t)) =$

$(x'_1+1, x'_2+1, x'_3+1, j)$ until one of the queues becomes empty for the first time, where (x_1, x_2, x_3, j) , $(x'_1, x'_2, x'_3, j) \in \mathbb{Z}_+^3 \times S_0$. The values of z_1^{max} , z_2^{max} and z_3^{max} can numerically be determined. For example, set $\lambda_1 = 0.1$, $\lambda_2 = 0.2$ and $\mu_1 = \mu_2 = \mu_3 = 1$; if $\lambda_3 = 0.3$, we have $z_1^{max} = 5.53$, $z_2^{max} = 2.77$ and $z_3^{max} = 1.85$ and if $\lambda_3 = 0.6$, we have $z_1^{max} = 7.77$, $z_2^{max} = 3.88$ and $z_3^{max} = 1.29$.

4 Concluding remarks

For analyzing asymptotics of the stationary distributions of multidimensional MMRRWs, the matrix analytic methods we developed in this paper are incomplete at least in two points. One is that, as mentioned in Remark 3.1, we can derive only a lower bound of the convergence parameter of the rate matrix. In order to improve this point, we must strengthen Lemma 2.4. The other is that we can not treat the case where a target model has boundaries like a multidimensional MMRRW, i.e., the case where the transition probability matrix and its block matrices are given in the same form as formula (1.1). In order to solve an asymptotic problem in such a case, we must strengthen our methods so that they can treat a nonnegative block tri-diagonal matrix with countably many phases given by

$$Q = \begin{pmatrix} B_0 & B_1 & & & \\ B_{-1} & A_0 & A_1 & & \\ & A_{-1} & A_0 & A_1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \quad (4.1)$$

In this paper, we considered only the case where $B_{-1} = A_{-1}$, $B_0 = A_0$ and $B_1 = A_1$. If we obtain, for this Q , a useful property corresponding to that given in Lemma 2.9, we can evaluate the directional asymptotic decay rates for the stationary distribution of a multidimensional MMRRW.

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A Convexity of the reciprocal of a convergence parameter

Let n be a positive integer and $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. According to Kingman [5], we say that a positive function $f(\mathbf{x})$ is superconvex in \mathbf{x} if $\log f(\mathbf{x})$ is convex in \mathbf{x} , and denote by \mathfrak{S}_n the class of all superconvex functions of n variables, together with the function identically zero. Note that, \mathfrak{S}_n is closed under addition, multiplication, raising to any positive power, and “lim sup” operation. Furthermore, a superconvex function is a convex function.

Let $F(\mathbf{x}) = (f_{ij}(\mathbf{x}), i, j \in \mathbb{Z}_+)$ be a matrix function each of whose elements belongs to the class \mathfrak{S}_n , i.e., for ever $i, j \in \mathbb{Z}_+$, $f_{i,j} \in \mathfrak{S}_n$. In Kingman [5], it is demonstrated that when $n = 1$ and $F(x)$ is a square matrix of a finite dimension, the maximum eigenvalue of $F(x)$ is a superconvex function in x . Analogously, we obtain the following lemma.

Lemma A.1. *For every $\mathbf{x} \in \mathbb{R}^n$, assume all iterates of $F(\mathbf{x})$ is finite and $F(\mathbf{x})$ is irreducible. Then, the reciprocal of the convergence parameter of $F(\mathbf{x})$, $\text{cp}(F(\mathbf{x}))^{-1}$, is superconvex in \mathbf{x} or identically zero.*

Proof. For $k \geq 0$, we denote by $f_{i,j}^{(k)}(\mathbf{x})$ the (i, j) -element of $F(\mathbf{x})^k$. First, we show that, for any $k \geq 1$ and for any $i, j \in \mathbb{Z}_+$, $f_{i,j}^{(k)}(\mathbf{x}) \in \mathfrak{S}_n$. It is obvious when $k = 1$. Suppose that it holds for k . Then, we have, for any $i, j \in \mathbb{Z}_+$,

$$f_{i,j}^{(k+1)}(\mathbf{x}) = \lim_{m \rightarrow \infty} \sum_{l=0}^m f_{i,l}^{(k)}(\mathbf{x}) f_{l,j}(\mathbf{x}), \quad (\text{A.1})$$

and this leads us to $f_{i,j}^{(k+1)}(\mathbf{x}) \in \mathfrak{S}_n$ since \mathfrak{S}_n is closed under addition, multiplication and “lim sup” (“lim”) operation. Therefore, for any $k \geq 1$, every element of $F(\mathbf{x})^n$ belongs to \mathfrak{S}_n .

Next, we note that, by Theorem 6.1 of Seneta [17], since $F(\mathbf{x})$ is irreducible, all elements of the power series $\sum_{k=0}^{\infty} z^k F(\mathbf{x})^k$ have the common convergence radius (convergence parameter), which is denoted by $\text{cp}(F(\mathbf{x}))$. By Cauchy-Hadamard theorem, we have, for any $i, j \in \mathbb{Z}_+$,

$$\text{cp}(F(\mathbf{x}))^{-1} = \limsup_{k \rightarrow \infty} (f_{i,j}^{(k)}(\mathbf{x}))^{1/k}, \quad (\text{A.2})$$

and this implies $\text{cp}(F(\mathbf{x}))^{-1} \in \mathfrak{S}_n$ since $(f_{i,j}^{(k)}(\mathbf{x}))^{1/k} \in \mathfrak{S}_n$ for any $k \geq 1$. \square

B Proof of Proposition 2.2

Proof. (i) For $n \geq 1$, $\mathcal{J}_{U,1,n}$ and $\mathcal{J}_{U,1,n}$ satisfy

$$\begin{aligned} \mathcal{J}_{D,1,n} &= \left\{ \mathbf{i}_{(n)} \in \mathbb{H}^n : \sum_{l=1}^k i_l \geq 0 \text{ for } k \in \{1, 2, \dots, n-2\}, \sum_{l=1}^{n-1} i_l = 0 \text{ and } i_n = -1 \right\} \\ &= \{(\mathbf{i}_{(n-1)}, -1) : \mathbf{i}_{(n-1)} \in \mathcal{J}_{n-1}\}, \\ \mathcal{J}_{U,1,n} &= \left\{ \mathbf{i}_{(n)} \in \mathbb{H}^n : i_1 = 1, \sum_{l=2}^k i_l \geq 0 \text{ for } k \in \{2, \dots, n-1\} \text{ and } \sum_{l=2}^n i_l = 0 \right\} \\ &= \{(1, \mathbf{i}_{(n-1)}) : \mathbf{i}_{(n-1)} \in \mathcal{J}_{n-1}\}, \end{aligned}$$

where $\mathbf{i}_{(n)} = (i_1, i_2, \dots, i_n)$. Hence, by Fubini's theorem, we have, for $i, j \in \mathbb{Z}_+$,

$$\begin{aligned} [G]_{i,j} &= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} [Q_{0,0}^{(n-1)}]_{i,k} [A_{-1}]_{k,j} = [NA_{-1}]_{i,j}, \\ [R]_{i,j} &= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} [A_1]_{i,k} [Q_{0,0}^{(n-1)}]_{k,j} = [A_1N]_{i,j}. \end{aligned}$$

(ii) We prove equation (2.3). For $n \geq 3$, $\mathcal{J}_{D,1,n}$ satisfies

$$\begin{aligned} \mathcal{J}_{D,1,n} &= \left\{ \mathbf{i}_{(n)} \in \mathbb{H}^n : i_1 = 0, \sum_{l=2}^k i_l \geq 0 \text{ for } k \in \{2, \dots, n-1\}, \sum_{l=2}^n i_l = -1 \right\} \\ &\quad \cup \left\{ \mathbf{i}_{(n)} \in \mathbb{H}^n : i_1 = 1, \sum_{l=2}^k i_l \geq -1 \text{ for } k \in \{2, \dots, n-1\}, \sum_{l=2}^n i_l = -2 \right\} \\ &= \{(0, \mathbf{i}_{(n-1)}) : \mathbf{i}_{(n-1)} \in \mathcal{J}_{D,1,n-1}\} \cup \{(1, \mathbf{i}_{(n-1)}) : \mathbf{i}_{(n-1)} \in \mathcal{J}_{D,2,n-1}\}, \end{aligned}$$

and $\mathcal{J}_{D,2,n}$ satisfies

$$\begin{aligned} \mathcal{J}_{D,2,n} &= \bigcup_{m=1}^{n-1} \left\{ \mathbf{i}_{(n)} \in \mathbb{H}^n : \sum_{l=1}^k i_l \geq 0 \text{ for } k \in \{1, 2, \dots, m-1\}, \sum_{l=1}^m i_l = -1, \right. \\ &\quad \left. \sum_{l=m+1}^k i_l \geq 0 \text{ for } k \in \{m+1, m+2, \dots, n-1\} \text{ and } \sum_{l=m+1}^n i_l = -1 \right\} \\ &= \bigcup_{m=1}^{n-1} \{(\mathbf{i}_{(m)}, \mathbf{i}_{(n-m)}) : \mathbf{i}_{(m)} \in \mathcal{J}_{D,1,m} \text{ and } \mathbf{i}_{(n-m)} \in \mathcal{J}_{D,1,m-n}\}. \end{aligned}$$

Hence, we have, for $n \geq 3$,

$$\begin{aligned} D^{(n)} &= A_0 D^{(n-1)} + A_1 \sum_{\mathbf{i}_{(n-1)} \in \mathcal{J}_{D,2,n-1}} A_{i_1} A_{i_2} \cdots A_{i_{n-1}} \\ &= A_0 D^{(n-1)} + A_1 \sum_{m=1}^{n-1} D^{(m)} D^{(n-m-1)}, \end{aligned}$$

and by Fubini's theorem, we obtain, for $i, j \in \mathbb{Z}_+$,

$$\begin{aligned} [G]_{i,j} &= [D^{(1)}]_{i,j} + \sum_{n=2}^{\infty} \sum_{k=0}^{\infty} [A_0]_{i,k} [D^{(n-1)}]_{k,j} \\ &\quad + \sum_{n=3}^{\infty} \sum_{m=1}^{n-2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} [A_1]_{i,k} [{}^m D^{(m)}]_{k,l} [D^{(n-m-1)}]_{l,j} \\ &= [A_{-1}]_{i,j} + [A_0 G]_{i,j} + [A_1 G^2]_{i,j}, \end{aligned}$$

where we use the fact that $D^{(1)} = A_1$ and $D^{(2)} = A_0 A_{-1} = A_0 D^{(1)}$. Equation (2.2) is analogously proved.

(iii) We prove equation (2.4). For $n \geq 1$, \mathcal{I}_n satisfies

$$\begin{aligned}
\mathcal{I}_n &= \left\{ \mathbf{i}_{(n)} \in \mathbb{H}^n : i_1 = 0, \sum_{l=2}^k i_l \geq 0 \text{ for } k \in \{2, \dots, n-1\}, \sum_{l=2}^n i_l = 0 \right\} \\
&\quad \cup \left\{ \mathbf{i}_{(n)} \in \mathbb{H}^n : i_1 = 1, \sum_{l=2}^k i_l \geq -1 \text{ for } k \in \{2, \dots, n-1\}, \sum_{l=2}^n i_l = -1 \right\} \\
&= \{(0, \mathbf{i}_{(n-1)}) : \mathbf{i}_{(n-1)} \in \mathcal{I}_{n-1}\} \\
&\quad \cup \bigcup_{m=2}^n \left\{ \mathbf{i}_{(n)} \in \mathbb{H}^n : i_1 = 1, \sum_{l=2}^k i_l \geq 0 \text{ for } k \in \{2, \dots, m-1\}, \sum_{l=2}^m i_l = -1, \right. \\
&\quad \left. \sum_{l=m+1}^k i_l \geq 0 \text{ for } k \in \{m+1, \dots, n-1\}, \sum_{l=m+1}^n i_l = 0 \right\} \\
&= \{(0, \mathbf{i}_{(n-1)}) : \mathbf{i}_{(n-1)} \in \mathcal{I}_{n-1}\} \\
&\quad \cup \left(\bigcup_{m=2}^n \{(1, \mathbf{i}_{(m-1)}, \mathbf{i}_{(n-m)}) : \mathbf{i}_{(m-1)} \in \mathcal{I}_{D,1,m-1}, \mathbf{i}_{(n-m)} \in \mathcal{I}_{n-m}\} \right).
\end{aligned}$$

Hence, we have, for $n \geq 1$,

$$Q_{0,0}^{(n)} = A_0 Q_{0,0}^{(n-1)} + \sum_{m=2}^n A_1 D^{(m-1)} Q_{0,0}^{(n-m)},$$

and by Fubini's theorem, we obtain, for $i, j \in \mathbb{Z}_+$,

$$\begin{aligned}
[N]_{i,j} &= \delta_{ij} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} [A_0]_{i,k} [Q_{0,0}^{(n-1)}]_{k,j} + \sum_{n=2}^{\infty} \sum_{m=2}^n \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} [A_1]_{i,k} [D^{(m-1)}]_{k,l} [Q_{0,0}^{(n-m)}]_{l,j} \\
&= \delta_{i,j} + [A_0 N]_{i,j} + [A_1 G N]_{i,j}.
\end{aligned}$$

This leads us to equation (2.4). □

C Proof of Theorem 3.1

Proof. Recall that, for $\mathbf{y}, \mathbf{y}' \in \mathbb{S}_+$,

$$\tilde{q}_{\mathbf{y}, \mathbf{y}'} = \mathbb{E} \left(\sum_{n=0}^{\sigma_+ - 1} 1(\mathbf{Y}_n = \mathbf{y}') \mid \mathbf{Y}_0 = \mathbf{y} \right), \quad (\text{C.1})$$

where σ_+ is a stopping time defined as $\sigma_+ = \inf\{n \geq 0; \mathbf{Y}_n \in \mathbb{S} \setminus \mathbb{S}_+\}$. This leads us to, for any $\mathbf{y} \in \mathbb{S}_+$,

$$\sum_{\mathbf{y}' \in \mathbb{S}_+} \tilde{q}_{\mathbf{y}, \mathbf{y}'} = \mathbb{E}(\sigma_+ \mid \mathbf{Y}_0 = \mathbf{y}). \quad (\text{C.2})$$

Hence, in order to prove the theorem, it suffices to show that $\mathbb{E}(\sigma_+ \mid \mathbf{Y}_0 = \mathbf{y})$ is finite for any $\mathbf{y} \in \mathbb{S}_+$.

To this end, we consider a 3D-MMRRW on the state space $\mathbb{S}_+ = \mathbb{Z}_+^3 \times S_0$; we denote it by $\{\hat{\mathbf{Y}}_n\} = \{(\hat{X}_{1,n}, \hat{X}_{2,n}, \hat{X}_{3,n}, \hat{J}_n)\}$. Let \mathbb{M}_3 be the set of all subsets of \mathbb{N}_3 , i.e.,

$$\mathbb{M}_3 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \mathbb{N}_3\}.$$

We use \mathbb{M}_3 as an index set. Exclusively divide the state space \mathbb{S}_+ into eight subsets: \mathbb{S}_+^α , $\alpha \in \mathbb{M}_3$, defined as

$$\mathbb{S}_+^\alpha = \{(x_1, x_2, x_3, j) \in \mathbb{S}_+; x_i \geq 1 \text{ for } i \in \alpha \text{ and } x_i = 0 \text{ for } i \in \mathbb{N}_3 \setminus \alpha\}.$$

For $\alpha, \beta \in \mathbb{M}_3$ and for $k, l, m \in \mathbb{H}$, denote by $P_{klm}^{\alpha, \beta}$ the matrix of one-step transition probabilities that the 3D-MMRRW $\{\hat{\mathbf{Y}}_n\}$ moves from a state in \mathbb{S}_+^α to a state in \mathbb{S}_+^β , i.e., for $(x_1, x_2, x_3, j) \in \mathbb{S}_+^\alpha$ and $(x_1 + k, x_2 + l, x_3 + m, j') \in \mathbb{S}_+^\beta$,

$$\left[P_{klm}^{\alpha, \beta} \right]_{jj'} = \mathbb{P}(\hat{\mathbf{Y}}_{n+1} = (x_1 + k, x_2 + l, x_3 + m, j') \mid \hat{\mathbf{Y}}_n = (x_1, x_2, x_3, j)).$$

The transition probability matrix of the 3D-MMRRW is composed of such block matrices. We assume the block matrices that are nonzero are given by

$$\begin{aligned} P_{klm}^{\mathbb{N}_3, \mathbb{N}_3} &= A_{klm}, \quad k, l, m \in \mathbb{H}, \quad P_{k,l,-1}^{\mathbb{N}_3, \{1,2\}} = A_{k,l,-1}, \quad k, l \in \mathbb{H}, \quad P_{-1,l,m}^{\mathbb{N}_3, \{2,3\}} = A_{-1,l,m}, \quad l, m \in \mathbb{H}, \\ P_{k,-1,m}^{\mathbb{N}_3, \{1,3\}} &= A_{k,-1,m}, \quad k, m \in \mathbb{H}, \quad P_{k,-1,-1}^{\mathbb{N}_3, \{1\}} = A_{k,-1,-1}, \quad k \in \mathbb{H}, \quad P_{-1,l,-1}^{\mathbb{N}_3, \{2\}} = A_{-1,l,-1}, \quad l \in \mathbb{H}, \\ P_{-1,-1,m}^{\mathbb{N}_3, \{3\}} &= A_{-1,-1,m}, \quad m \in \mathbb{H}, \quad P_{-1,-1,-1}^{\mathbb{N}_3, \emptyset} = A_{-1,-1,-1}, \\ P_{-1,0,0}^{\{1,2\}, \{1,2\}} &= P_{0,-1,0}^{\{1,2\}, \{1,2\}} = P_{0,-1,0}^{\{1,2\}, \{1\}} = P_{-1,0,0}^{\{1,2\}, \{2\}} = \frac{1}{2s_0} E, \\ P_{0,-1,0}^{\{2,3\}, \{2,3\}} &= P_{0,0,-1}^{\{2,3\}, \{2,3\}} = P_{0,0,-1}^{\{2,3\}, \{2\}} = P_{0,-1,0}^{\{2,3\}, \{3\}} = \frac{1}{2s_0} E, \\ P_{-1,0,0}^{\{1,3\}, \{1,3\}} &= P_{0,0,-1}^{\{1,3\}, \{1,3\}} = P_{0,0,-1}^{\{1,3\}, \{1\}} = P_{-1,0,0}^{\{1,3\}, \{3\}} = \frac{1}{2s_0} E, \\ P_{-1,0,0}^{\{1\}, \{1\}} &= P_{-1,0,0}^{\{1\}, \emptyset} = \frac{1}{s_0} E, \quad P_{0,-1,0}^{\{2\}, \{2\}} = P_{0,-1,0}^{\{2\}, \emptyset} = \frac{1}{s_0} E, \quad P_{0,0,-1}^{\{3\}, \{3\}} = P_{0,0,-1}^{\{3\}, \emptyset} = \frac{1}{s_0} E, \\ P_{0,0,0}^{\emptyset, \emptyset} &= \frac{1}{s_0} E, \end{aligned}$$

where E is an $s_0 \times s_0$ matrix of 1's. Then, the 3D-MMRRW $\{\hat{\mathbf{Y}}_n\}$ becomes a reducible Markov chain with a unique irreducible class \mathbb{S}_+^\emptyset . We define a stopping time $\hat{\sigma}_+$ as

$$\hat{\sigma}_+ = \inf\{n \geq 0; \hat{\mathbf{Y}}_n \in \mathbb{S}_+^\emptyset\}.$$

On the subspace $\mathbb{S}_+^{\mathbb{N}_3, \mathbb{N}_3}$, the 3D-MMRRW $\{\hat{\mathbf{Y}}_n\}$ behaves just like the original 3D-MMRW $\{\mathbf{Y}_n\}$ in a stochastic sense. Furthermore, after entering any state in $\mathbb{S}_+^{\{1,2\}} \cup \mathbb{S}_+^{\{2,3\}} \cup \mathbb{S}_+^{\{1,3\}}$, the 3D-MMRRW moves toward $\mathbb{S}_+^{\{1\}}$, $\mathbb{S}_+^{\{2\}}$ or $\mathbb{S}_+^{\{3\}}$, without returning to $\mathbb{S}_+^{\mathbb{N}_3, \mathbb{N}_3}$; after entering any state in $\mathbb{S}_+^{\{1\}} \cup \mathbb{S}_+^{\{2\}} \cup \mathbb{S}_+^{\{3\}}$, it moves just toward the unique irreducible class \mathbb{S}_+^\emptyset , without returning to $\mathbb{S}_+^{\mathbb{N}_3, \mathbb{N}_3} \cup \mathbb{S}_+^{\{1,2\}} \cup \mathbb{S}_+^{\{2,3\}} \cup \mathbb{S}_+^{\{1,3\}}$. Hence, we obtain for $(x_1, x_2, x_3, j) \in \mathbb{S}_+$ and for $n \geq 0$,

$$\mathbb{P}(\sigma_+ > n \mid \mathbf{Y}_0 = (x_1, x_2, x_3, j)) \leq \mathbb{P}(\hat{\sigma}_+ > n \mid \hat{\mathbf{Y}}_0 = (x_1 + 1, x_2 + 1, x_3 + 1, j)), \quad (\text{C.3})$$

and this leads us to

$$\mathbb{E}(\sigma_+ | \mathbf{Y}_0 = (x_1, x_2, x_3, j)) \leq \mathbb{E}(\hat{\sigma}_+ | \hat{\mathbf{Y}}_0 = (x_1 + 1, x_2 + 1, x_3 + 1, j)). \quad (\text{C.4})$$

Hence, in order to prove the theorem, it suffices to show that $\mathbb{E}(\hat{\sigma}_+ | \hat{\mathbf{Y}}_0 = \mathbf{y})$ is finite for any $\mathbf{y} \in \mathbb{S}^{\mathbb{N}_3, \mathbb{N}_3}$.

The 3D-MMRRW $\{\hat{\mathbf{Y}}_n\}$ has seven induced Markov chains, which we denote by \mathcal{L}^α , $\alpha \in \mathbb{M}_3 \setminus \{\emptyset\}$; \mathcal{L}^\emptyset is $\{\hat{\mathbf{Y}}_n\}$ itself. If an induced Markov chain is stable (has a unique stationary distribution), we can define a mean drift vector with respect to the induced Markov chain (for induced Markov chains and their mean drift vectors, see Fayolle et al. [2] and Ozawa [15]). For $\alpha \in \mathbb{M}_3 \setminus \{\emptyset\}$, we denote by \mathbf{a}^α the mean drift vector with respect to the induced Markov chain \mathcal{L}^α . Using the mean drift vectors, we can determine that $\{\hat{\mathbf{Y}}_n\}$ is stable or not. $\mathcal{L}^{\mathbb{N}_3}$ is a finite Markov chain governed by the stochastic matrix A_{***} . Since A_{***} is irreducible, $\mathcal{L}^{\mathbb{N}_3}$ is positive recurrent and the mean drift vector $\mathbf{a}^{\mathbb{N}_3}$ is equal to $\mathbf{a} = (a_1, a_2, a_3)$, given by expression (3.2). Note that, under Assumption 3.1, every element of \mathbf{a} is nonzero and at least one element of \mathbf{a} is negative.

$\mathcal{L}^{\{1,2\}}$ is a QBD process on the state space $\mathbb{Z}_+ \times S_0$ whose transition probability matrix $P^{\{1,2\}}$ is given in block form by

$$P^{\{1,2\}} = \begin{pmatrix} \frac{1}{s_0}E & O & & & \\ A_{**, -1} & A_{**, 0} & A_{**, 1} & & \\ & A_{**, -1} & A_{**, 0} & A_{**, 1} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

This transition matrix shows that $\mathcal{L}^{\{1,2\}}$ is reducible and has just one irreducible class $\{0\} \times S_0$. Hence, if $\mathcal{L}^{\{1,2\}}$ is stable, its unique stationary distribution is given in block form by $(\frac{1}{s_0}\mathbf{e}_{s_0}^\top \quad \mathbf{0}_{s_0}^\top \quad \mathbf{0}_{s_0}^\top \quad \dots)$, where \mathbf{e}_{s_0} is an $s_0 \times 1$ vector of 1's and $\mathbf{0}_{s_0}$ an $s_0 \times 1$ vector of 0's; the mean drift vector $\mathbf{a}^{\{1,2\}}$ is given by

$$\mathbf{a}^{\{1,2\}} = \left(\frac{1}{s_0}\mathbf{e}_{s_0}^\top \left(-\frac{1}{2s_0}E \right) \mathbf{e}_{s_0}, \frac{1}{s_0}\mathbf{e}_{s_0}^\top \left(-\frac{1}{2s_0}E \right) \mathbf{e}_{s_0}, 0 \right) = \left(-\frac{1}{2}, -\frac{1}{2}, 0 \right). \quad (\text{C.5})$$

In the same manner, we see that if $\mathcal{L}^{\{2,3\}}$ is stable, then $\mathbf{a}^{\{2,3\}} = (0, -\frac{1}{2}, -\frac{1}{2})$ and if $\mathcal{L}^{\{1,3\}}$ is stable, then $\mathbf{a}^{\{1,3\}} = (-\frac{1}{2}, 0, -\frac{1}{2})$. $\mathcal{L}^{\{1\}}$ is a 2D-MMRRW on the state space $\mathbb{Z}_+^2 \times S_0$ whose transition probability matrix $P^{\{1\}}$ is given in block form by

$$P^{\{1\}} = \begin{pmatrix} B_0^{\{1\}} & O & & & \\ A_{-1}^{\{1\}} & A_0^{\{1\}} & A_1^{\{1\}} & & \\ & A_{-1}^{\{1\}} & A_0^{\{1\}} & A_1^{\{1\}} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad B_0^{\{1\}} = \begin{pmatrix} \frac{1}{s_0}E & O & & & \\ \frac{1}{2s_0}E & \frac{1}{2s_0}E & O & & \\ & \frac{1}{2s_0}E & \frac{1}{2s_0}E & O & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$A_i^{\{1\}} = \begin{pmatrix} \frac{1-\delta_{i,1}}{2s_0}E & O & & & \\ A_{*, i, -1} & A_{*, i, 0} & A_{*, i, 1} & & \\ & A_{*, i, -1} & A_{*, i, 0} & A_{*, i, 1} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, i \in \mathbb{H},$$

where if $i = 1$, then $\delta_{i,1} = 1$, otherwise $\delta_{i,1} = 0$. This $P^{\{1\}}$ shows that $\mathcal{L}^{\{1\}}$ is reducible and has just one irreducible class $\{(0, 0)\} \times S_0$. Hence, if $\mathcal{L}^{\{1\}}$ is stable, its unique stationary distribution is

given in block form by $\begin{pmatrix} \boldsymbol{\pi}_0^{\{1\}} & \mathbf{0}^\top & \mathbf{0}^\top & \cdots \end{pmatrix}$, where $\boldsymbol{\pi}_0^{\{1\}} = (\frac{1}{s_0} \mathbf{e}_{s_0}^\top \quad \mathbf{0}^\top \quad \mathbf{0}^\top \quad \cdots)$, and the mean drift vector $\mathbf{a}^{\{1\}}$ by

$$\mathbf{a}^{\{1\}} = \left(\frac{1}{s_0} \mathbf{e}_{s_0}^\top \left(-\frac{1}{s_0} E \right) \mathbf{e}_{s_0}, 0, 0 \right) = (-1, 0, 0). \quad (\text{C.6})$$

In the same manner, we see that if $\mathcal{L}^{\{2\}}$ is stable, then $\mathbf{a}^{\{2\}} = (0, -1, 0)$ and if $\mathcal{L}^{\{3\}}$ is stable, then $\mathbf{a}^{\{3\}} = (0, 0, -1)$.

As a result, by the criteria obtained in Section 3.5.3 of Ozawa [15], we know that the 3D-MMRRW $\{\hat{\mathbf{Y}}_n\}$ is stable, and this implies that the expected first passage time to the unique irreducible class \mathbb{S}_+^\emptyset is finite, i.e., for any $\mathbf{y} \in \mathbb{S}_+ \setminus \mathbb{S}_+^\emptyset$, $\mathbb{E}(\hat{\sigma}_+ | \hat{\mathbf{Y}}_0 = \mathbf{y}) < \infty$. Now, finiteness of each element of \tilde{Q} is obvious, and this completes the proof. \square